

Восстановление квадратичного дифференциального пучка с целыми функциями в краевых условиях¹

М. А. Кузнецова (Саратов, Россия)

kuznetsovama@info.sgu.ru

В работе изучается обратная спектральная задача для квадратичного пучка операторов Штурма–Лиувилля с сингулярными коэффициентами и целыми функциями в краевых условиях. Доказывается, что для восстановления пучка достаточно части спектра, если по ней можно построить полную функциональную последовательность. Полученные результаты применяются в исследовании неполной обратной задачи.

Ключевые слова: обратные спектральные задачи, дифференциальные пучки, аналитическая зависимость от спектрального параметра, сингулярные коэффициенты, теорема единственности, неполные обратные задачи.

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On recovering quadratic differential pencils with entire functions in the boundary conditions¹

M. A. Kuznetsova (Saratov, Russia)

kuznetsovama@info.sgu.ru

In this work, we study an inverse spectral problem for quadratic pencils of Sturm–Liouville operators with singular coefficients and entire functions in the boundary conditions. It is proved that a part of the spectrum is sufficient for recovering the pencil if this part generates a complete functional system. As well, we apply the obtained results to studying a partial inverse problem.

Keywords: inverse spectral problems, differential pencils, analytical dependence on the spectral parameter, singular coefficients, uniqueness theorem, partial inverse problems.

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Introduction

We study an inverse spectral problem of recovering the coefficients in the quadratic pencil

$$-y'' + q(x)y + 2\lambda p(x)y = \lambda^2 y, \quad x \in (0, \pi), \quad (1)$$

where $p \in L_2(0, \pi)$ and $q \in W_2^{-1}(0, \pi)$. The latter means $q = \sigma'$ in the sense of distributions for some $\sigma \in L_2(0, \pi)$. In the case of singular coefficients,

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inverse spectral problems of recovering the quadratic pencils were studied in the works [1, 2]. We rewrite and study equation (1) in the form with quasiderivative $y^{[1]} := y' - \sigma y$, as it was done in [1, 2].

We consider the following boundary conditions:

$$y(0) = 0, \quad f_1(\lambda)y^{[1]}(\pi) + f_2(\lambda)y(\pi) = 0, \quad (2)$$

where $f_1(\lambda)$ and $f_2(\lambda)$ are known entire functions. The reason for using conditions (2) is that they allow to unify studying partial inverse problems. In [3], it was shown that partial inverse problems for Sturm–Liouville operators of various classes reduce to recovering their potentials by subspectra of the boundary value problems with conditions (2).

Our aim is to study the inverse problem of recovering the pencil (1) with the boundary conditions (2) by a subspectrum $\{\lambda_n\}_{n \geq 1}$ and some number giving information on p :

Inverse problem 1. Given $\{\lambda_n\}_{n \geq 1}$ and $(\omega_0 \bmod 1)$, recover the coefficients p and q .

Here, we apply the technique developed in [3] to studying this inverse problem. We find the conditions on a part of the spectrum under which Inverse Problem 1 has a unique solution. These conditions include the completeness of certain functional sequences. Under them, we obtain a uniqueness theorem for Inverse Problem 1. Then, we provide an example of the partial inverse problem to which these results are applicable.

Preliminaries

Let $S(x, \lambda)$ be the solution of equation (1) satisfying the initial conditions $S(0, \lambda) = 0$, $S^{[1]}(0, \lambda) = 1$. The following representations hold, see [2]:

$$\left. \begin{aligned} S(\pi, \lambda) &= \frac{\sin \pi(\lambda - \omega_0)}{\lambda} + \frac{1}{\lambda} \int_{-\pi}^{\pi} \mathcal{K}(t) \exp(i\lambda t) dt, \\ S^{[1]}(\pi, \lambda) &= \cos \pi(\lambda - \omega_0) + \int_{-\pi}^{\pi} \mathcal{N}(t) \exp(i\lambda t) dt, \end{aligned} \right\} \quad (3)$$

where $\omega_0 = \frac{1}{\pi} \int_0^\pi p(s) ds$ and $\mathcal{K}, \mathcal{N} \in L_2(-\pi, \pi)$. As the piece of the input data for Inverse Problem 1, we take the fractional part $(\omega_0 \bmod 1)$ of ω_0 .

First, we note that a number λ is an eigenvalue of the boundary value problem (1), (2) if and only if it is a zero of the characteristic function

$$\Delta(\lambda) = f_1(\lambda)S^{[1]}(\pi, \lambda) + f_2(\lambda)S(\pi, \lambda). \quad (4)$$

Consider a sequence $\{\lambda_k\}_{k \geq 1}$ such that $\Delta(\lambda_k) = 0$ and each λ_k occurs in the sequence not more times than its multiplicity as zero of $\Delta(\lambda)$. We call such sequence $\{\lambda_k\}_{k \geq 1}$ a subspectrum.

Put $\lambda_0 := 0$ and introduce the notations

$$\mathbb{S}_\lambda = \{n \geq 0: \lambda_n \neq \lambda_k \quad \forall k: 0 \leq k < n\}, \quad m_{\lambda,n} = \#\{k \geq 0: \lambda_k = \lambda_n\}.$$

We assume that equal numbers in the sequence $\{\lambda_k\}_{k \geq 0}$ follow each other:

$$\lambda_n = \lambda_{n+1} = \dots = \lambda_{n+m_{\lambda,n}-1}, \quad n \in \mathbb{S}_\lambda.$$

Consider the Hilbert space of complex-valued vector-functions

$$\mathcal{H} = L_2(-\pi, \pi) \oplus L_2(-\pi, \pi).$$

For $g = [g_1, g_2]$ and $h = [h_1, h_2]$, the scalar product and the norm in \mathcal{H} are given by the formulae

$$(g, h)_{\mathcal{H}} = \int_{-\pi}^{\pi} [\overline{g_1(t)}h_1(t) + \overline{g_2(t)}h_2(t)] dt, \quad \|h\|_{\mathcal{H}} = \sqrt{(h, h)}.$$

In particular, we have $u(t) := [\overline{\mathcal{N}}(t), \overline{\mathcal{K}}(t)] \in \mathcal{H}$.

Let us introduce the notations

$$v(t, \lambda) = [\lambda f_1(\lambda)e(t, \lambda), f_2(\lambda)e(t, \lambda)], \quad e(x, \lambda) = \exp(i\lambda x).$$

For $n \in \mathbb{S}_\lambda$ and $\nu = \overline{0, m_{\lambda,n} - 1}$, we denote

$$v_{n+\nu}(t) = \begin{cases} v^{<\nu>}(t, \lambda_n), & n + \nu > 0, \\ [0, 1], & n = \nu = 0, \end{cases} \quad f^{<j>}(\lambda) = \frac{1}{j!} \frac{d^j}{dz^j} f(z) \Big|_{z=\lambda}.$$

Results

Let $L(p, q)$ be the boundary value problem (1), (2) with arbitrary coefficients $p \in L_2(0, \pi)$ and $q \in W_2^{-1}(0, \pi)$. Along with $L(p, q)$, we consider another problem $L(\tilde{p}, \tilde{q})$ of the same form but with other coefficients $\tilde{p} \in L_2(0, \pi)$ and $\tilde{q} \in W_2^{-1}(0, \pi)$. Let us agree that, if a symbol α denotes an object related to p and q , then the symbol $\tilde{\alpha}$ with tilde will denote the analogous object related to \tilde{p} and \tilde{q} .

Theorem 1. *Suppose that the sequence $\{v_n\}_{n=0}^{\infty}$ constructed by $\{\lambda_n\}_{n=1}^{\infty}$ satisfies the following condition:*

(C) *The sequence $\{v_k\}_{k=0}^{\infty}$ is complete in \mathcal{H} .*

Then, the equalities $\{\lambda_n\}_{n=1}^{\infty} = \{\tilde{\lambda}_n\}_{n=1}^{\infty}$ and $(\omega_0 \bmod 1) = (\tilde{\omega}_0 \bmod 1)$ yield $p \equiv \tilde{p}$, $q \equiv \tilde{q}$.

The proof is based on obtaining explicit formulae for the coefficients $(u, v_k)_{\mathcal{H}}$ after substituting (3) into (4). Then, we determine the functions

\mathcal{N} and \mathcal{K} uniquely and reduce Inverse Problem 1 to the inverse problem studied in [2].

In some cases, condition (C) is more difficult to verify than two independent conditions imposed on the subspectrum and the pair of the entire functions in (2). The following theorem contains such conditions.

Theorem 2. *Under the following two assumptions, (C) is fulfilled:*

(S) *For $n \in \mathbb{N}$, the functions $f_1(\lambda)$ and $f_2(\lambda)$ do not vanish simultaneously in λ_n .*

(C2) *The functional sequence $\{e^{\langle \nu \rangle}(t, \lambda_n)\}_{n \in \mathbb{S}_\lambda, v=0, m_{\lambda, n}-1}$ is complete in $L_2(-2\pi, 2\pi)$.*

Consider the boundary value problem

$$-y'' + q(x)y + 2\lambda p(x)y = \lambda^2 y, \quad x \in (0, 2\pi), \quad (5)$$

$$y(0) = y(2\pi) = 0, \quad (6)$$

where $p \in L_2(0, 2\pi)$ and $q \in W_2^{-1}(0, 2\pi)$. The eigenvalues $\{\mu_k\}_{k \in \mathbb{Z}_0}$ of the boundary value problem (5), (6) satisfy the asymptotics

$$\mu_k = \frac{k}{2} + \frac{1}{2\pi} \int_0^{2\pi} p(t) dt + \varkappa_k, \quad \{\varkappa_k\}_{k \in \mathbb{Z}_0} \in \ell_2.$$

We assume that the coefficients p and q are known on $(\pi, 2\pi)$. In [4], in the regular case $p \in W_2^1(0, 2\pi)$ and $q \in L_2(0, 2\pi)$, the following inverse problem was studied:

Inverse Problem 2. Given $\{\mu_k\}_{k \in \mathbb{Z}_0}$ along with p and q on $(\pi, 2\pi)$, recover the coefficients p and q on the interval $(0, \pi)$.

For this inverse problem, a uniqueness theorem was obtained, see [4, Theorem 2]. By reducing Inverse Problem 2 to Inverse Problem 1, we generalize the mentioned result to the case when $p \in L_2(0, 2\pi)$ and $q \in W_2^{-1}(0, 2\pi)$.

Let us introduce the solution $\varphi(x, \lambda)$ of (5) satisfying the initial conditions

$$\varphi(2\pi, \lambda) = 0, \quad \varphi^{[1]}(2\pi, \lambda) = 1.$$

A number λ is an eigenvalue of (5), (6) if and only if

$$\varphi^{[1]}(\pi, \lambda)S(\pi, \lambda) - S^{[1]}(\pi, \lambda)\varphi(\pi, \lambda) = 0.$$

It is clear that $\varphi(\pi, \lambda)$ and $\varphi^{[1]}(\pi, \lambda)$ are known entire functions. Then, the eigenvalues $\{\mu_k\}_{k \in \mathbb{Z}_0}$ of (5), (6) coincide with the eigenvalues of (1), (2), where we put $f_1(\lambda) = \varphi(\pi, \lambda)$ and $f_2(\lambda) = \varphi^{[1]}(\pi, \lambda)$. This fact and the previous results allow us to obtain a uniqueness theorem.

Theorem 3. *Let $q = \tilde{q}$ in $W_2^{-1}(\pi, 2\pi)$, $p = \tilde{p}$ in $L_2(\pi, 2\pi)$, and $\{\mu_k\}_{k \in \mathbb{Z}_0} = \{\tilde{\mu}_k\}_{k \in \mathbb{Z}_0}$. Then, the identities $q \equiv \tilde{q}$ and $p \equiv \tilde{p}$ hold on $(0, 2\pi)$.*

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