УДК 517.9

О ФУНДАМЕНТАЛЬНЫХ РЕШЕНИЯХ РАЗНОСТНЫХ УРАВНЕНИЙ В КОНУСАХ ЦЕЛОЧИСЛЕННОЙ РЕШЕТКИ

С. Чандрагири, А. П. Ляпин (Красноярск, Россия)
srilathasami66@gmail.com, APLyapin@sfu-kras.ru

В работе рассмотрена задача Коши для многомерного разностного уравнения с постоянными коэффициентами в конусах целочисленной решетки. При помощи метода производящих функций доказана ее разрешимость и получено соотношение между производящими функциями решения задачи Коши и ее начальными данными. Как следствие, решение задачи Коши выражено через ее начальные данные и фундаментальное решение.

Ключевые слова: решеточные пути, целочисленный конус, разностное уравнение, производящая функция.

ON FUNDAMENTAL SOLUTIONS TO DIFFERENCE EQUATIONS IN LATTICE CONES

S. Chandragiri, A. P. Lyapin (Krasnoyarsk, Russia)
srilathasami66@gmail.com, APLyapin@sfu-kras.ru

We consider the Cauchy problem for a multidimensional difference equation with constant coefficients in the cones of an integer lattice. Using the method of generating functions, its solvability is proved and the relation between the generating functions of the solution to the Cauchy problem and its initial data is obtained. As a result, the solution to the Cauchy problem is expressed through its initial data and fundamental solution.

Keywords: lattice paths, lattice cone, difference equation, generating function.

On complex valued functions $f : \mathbb{Z}^n \to \mathbb{C}$ we define the shift operator

$$\delta_j : f(x_1, \ldots, x_j, \ldots, x_n) \mapsto f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n)$$

and polynomial difference operator

$$P(\delta) = \sum_{\omega \in \Omega} c_\omega \delta^\omega,$$

where $\Omega \subset \mathbb{Z}^n$ is a finite set of points of an n-dimensional lattice, $\delta^\omega = \delta_1^{\omega_1} \cdot \ldots \cdot \delta_n^{\omega_n}$ and $c_\omega \in \mathbb{C}$ are the coefficients of the difference operator.

Let $\alpha^1, \ldots, \alpha^N$ be the set of vectors $\alpha^j = (\alpha_1^j, \ldots, \alpha_n^j) \in \mathbb{Z}^n, j = 1, \ldots, N$, and K is a lattice cone spanned by these vectors

$$K = \{x \in \mathbb{Z}^n : x = \lambda_1 \alpha^1 + \cdots + \lambda_N \alpha^N, \lambda_i \in \mathbb{Z}_{\geqslant}, i = 1, \ldots, N\}.$$

We assume that cone K is *pointed*, which means it does not contain any line or, equivalently, lies in an open half-space of $\mathbb{R}^n$. We also define a relation $\underset{K}{\geqslant}$ as follows: $u \underset{K}{\geqslant} \nu \Leftrightarrow u - \nu \in K$ for any $u, \nu \in K$. Denote $m = \alpha^1 + \cdots + \alpha^m, c_0 = 1, \alpha^0 = (0, \ldots, 0)$ and formulate the problem.

The Cauchy Problem. Find a solution to the difference equation

$$\sum_{j=0}^{N} c_j f(x - \alpha^j) = g(x), \quad x \underset{K}{\geqslant} 0, \tag{1}$$

which coincides with the given function $\varphi(x)$ on the set $X_0 = \{x \in K : x \underset{K}{\ngeqslant} m\}$:

$$f(x) = \varphi(x), \quad x \in X_0. \tag{2}$$

We use equation (1) with initial data (2) to describe a wide class problems of enumerative combinatorial analysis including lattice paths problems (the Dyck, Motzkin and Schröder paths, generalized lattice paths. See [1], [3], [4]).

The fact that cone K is pointed allows us to prove the solvability of problem (1)–(2) using the method of generating functions, and namely, correctly define on the set of (formal) power series

$$F(z) = \sum_{x \underset{K}{\geqslant} 0} f(x) z^x$$

the structure of a ring, which will be denoted by $\mathbb{C}_K[[z]]$. Additionally we will denote $F_m(z) = \sum\limits_{x \underset{K}{\geqslant} m} f(x) z^x$.

Using the method of generating functions, the solvability of problem (1)–(2) was proved, namely, there is a formula in which generating function () is expressed in terms of generating functions $\Phi_m(z) = F(x) - F_m(z)$ and $G_m(z) = \sum\limits_{x \underset{K}{\geqslant} m} g(x) z^x$ of the initial data $\varphi(x)$ and the right-hand side $g(x)$ of equation (1) respectively.

Теорема 1. *The generating function $F(z)$ of a solution to the difference equation* (1) *with initial data* (2) *is represented as*

$$F(z) = \frac{1}{P(z^{-1})} \Big(\sum_{j=0}^{N} c_j z^{\alpha^j} \Phi_{m-\alpha^j}(z) + G_m(z) \Big),$$

where $z^{-1} = (z_1^{-1}, \ldots, z_n^{-1}), P(z) = \sum\limits_{j=0}^{N} c_j z^{-\alpha^j}$ is a characteristic polynomial of (1).

Further, using the concept of the fundamental solution $\mathcal{P}(x)$ of the problem (1)–(2) yields a formula expressing $f(x)$ in terms of $\varphi(x), g(x)$ and $\mathcal{P}(x)$.

A fundamental solution $\mathcal{P}(x)$ (see [2]) is a solution such that

$$\sum_{j=0}^{N} c_j \mathcal{P}(x-\alpha^j) = \delta_0(x), \quad x \in K,$$

where $\delta_0(x)$ is the Kronecker symbol.

We can obtain the fundamental solution $\mathcal{P}(x)$ by expanding function $P^{-1}(z^{-1})$ in a Laurent series as follows

$$\frac{1}{P(z^{-1})} = \sum_{x \underset{K}{\geqslant} 0} \mathcal{P}(x) z^x.$$

It allows us to find a solution to the Cauchy problem in terms of its initial data and fundamental solution.

Теорема 2. *A solution to the difference equation* (1) *with initial data* (2) *is given as follows:*

$$f(x) = \sum_{0 \underset{K}{\leqslant} y \underset{K}{\leqslant} x} \mathcal{P}(x-y)\tau(y),$$

where

$$\tau(y) = \begin{cases} \sum\limits_{j=0}^{N} c_j \varphi(y-\alpha^j), & \text{if } y \underset{K}{\not\geqslant} m; \\ g(y), & \text{if } y \underset{K}{\geqslant} m. \end{cases}$$

Example. Let $\alpha^1 = (2,-1), \alpha^2 = (-1,2)$ be a column vectors, we let K denote the cone K spanned by the vectors $K = \langle \alpha^1, \alpha^2 \rangle, s = \alpha^1 + \alpha^2 = (1,1)$.

We consider the two dimensional difference equation

$$f(x,y) - f(x-2,y+1) - f(x+1,y-2) = 0, \tag{3}$$

its characteristic polynomial $P(z,w) = 1 - z^{-2}w^1 - z^1 w^{-2}$.

According to Theorem 2, a solution to this difference equation is

$$f(x_1,x_2) = \sum_{0 \underset{K}{\leqslant} y \underset{K}{\leqslant} x} \mathcal{P}(x_1-y_1, x_2-y_2)\tau(y_1,y_2),$$

where $\tau(y_1,y_2) = \begin{cases} \varphi(y_1,y_2) - \varphi(y_1-2,y_2+1) - \varphi(y_1+1,y_2-2), \\ \qquad\qquad \text{if } (y_1,y_2) \not\geqslant (1,1); \\ 0, \ \text{if } (y_1,y_2) \geqslant (1,1). \end{cases}$

To find a fundamental solution we will expand the characteristic polynomial $P(z,w)$ into a series as follows:

$$\frac{1}{(1-z^2w^{-1}-z^{-1}w^2)} = \sum_{k=0}^{\infty}(z^2w^{-1}+z^{-1}w^2)^k =$$

$$= \sum_{(k_1,k_2)\underset{K}{\geqslant}0} \frac{(k_1+k_2)!}{\left(\frac{2k_1+k_2}{3}\right)!\left(\frac{k_1+2k_2}{3}\right)!} z^{k_1}w^{k_2}.$$

Consequently,

$$\mathcal{P}(k_1,k_2) = \frac{(k_1+k_2)!}{\left(\frac{2k_1+k_2}{3}\right)!\left(\frac{k_1+2k_2}{3}\right)!}.$$

Finally, we have the solution for difference equation (3) with arbitrary initial data

$$f(x_1,x_2) = \mathcal{P}(x_1,x_2)\varphi(0,0)+$$
$$\sum_{t=1}^{x_1}\mathcal{P}(x_1-2t,x_2+t)(\varphi(2t,-t)-\varphi(2t-2,-t+1))+$$
$$+\sum_{t=1}^{x_2}\mathcal{P}(x_1+t,x_2-2t)(\varphi(-t,2t)-\varphi(-t+1,2t-2)).$$

In case of lattice paths, $\varphi(2t,-t)-\varphi(2t-2,-t+1)=0$ for $t\geqslant 1$, $\varphi(-t,2t)-\varphi(-t+1,2t-2)=0$ for $t\geqslant 1$, and $\varphi(0,0)=1$, then we get $f(x_1,x_2)=\mathcal{P}(x_1,x_2)$.

REFERENCES

[1] *Bousquet Mélou M., Petkovšek M.* Linear recurrences with constant coefficients: The multivariate case // Discrete Math. 2000. Vol. 225. P. 51–75.

[2] *Leinartas E. K.* Multiple Laurent series and fundamental solutions of linear difference equations // Siberian Mathematical Journal. 2007. Vol. 48, № 2. P. 268–272.

[3] *Lyapin A. P., Chandragiri S.* Generating functions for vector partitions and a basic recurrence relation // Journal of Difference Equations and Applications. 2019. Vol. 25, № 7. P. 1052–1061.

[4] *Chandragiri S.* Difference equations and generating functions for some lattice path problems // Journal of Siberian Federal University, Mathematics & Physics. 2019. Vol. 12, № 5. P. 551–559.

[5] *Leinartas E. K., Lyapin A. P.* On the rationality of multidimensional recursive series // Journal of Siberian Federal University, Mathematics & Physics. 2009. Vol. 4, № 2. P. 449–455.

[6] *Stanley R.* Enumerative combinatorics. Cambridge : Cambridge Univ. Press, 1999.

[7] *Leinartas E. K., Nekrasova T. I.* Constant coefficient linear difference equations on the rational cones of the integer lattice // Siberian Mathematical Journal. 2016. Vol. 57, № 2. P. 98–112.