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A UNIQUENESS THEOREM OF THE INVERSE PROBLEM FOR A CLASS THE STURM– LIOUVILLE PROBLEM

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In the present paper , we study inverse problem of scattering theory for Sturm-Liouville operator on the half-axis $[0,\infty)$ with spectral paramater in the boundary condition for second order of differantial equation .We define the kernel function and determine the scattering data uniquely .

Keywords: Sturm–Liouville operator, scattering function, scattering data, Gelfand-Levitan-Marchenko main equation..

Introduction

We consider inverse problem of scattering theory for the Sturm–Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \tag{1}$$

on the semi-axis $[0,\infty)$ containing a spectral parameter in the boundary condition

$$y'(0) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y(0) = 0. \tag{2}$$

Here λ is a spectral parameter, α_i $(i = 0, 1, 2.)$ are real numbers that satisfy certain conditions. $q(x)$ is a real valued function satisfying the condition

$$\int_0^\infty (1 + x)\,|q(x)|\,dx < \infty. \tag{3}$$

It is well known (see [1]) that for all λ from the half-line Eq. (1) has the solution

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt. \tag{4}$$

The kernel $K(x,t)$ satisfies the inequality

$$K(x,t) \le \frac{1}{2}\sigma(\frac{x+t}{2})\exp\left\{\sigma_1(x) - \sigma_1(\frac{x+t}{2})\right\}. \tag{5}$$

The inverse problem of scattering data without any spectral parameter in boundary condition was solved in $[1,2]$. The many spectral properties of the boundary value problems were investigated with different methods by the many authors in [1–7].

In this work we prove the uniqueness of the solution of the inverse problem of scattering theory on the half line for the boundary — value problem (1)–(2) by using defined the sacttering data of the problem and its properties.

With the above preliminaries provided, we have the following lemmas and theorems.

Lemma 1. *For all* $\lambda \neq 0$*, the identity is valid*

$$\frac{2i\lambda w(x,\lambda)}{E(\lambda)} = e(x,-\lambda) - S(\lambda)e(x,\lambda), \tag{6}$$

where

$$S(\lambda) = \frac{E_1(\lambda)}{E(\lambda)}, \tag{7}$$

$$E(\lambda) = e'(0,\lambda) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)e(0,\lambda),$$

$$E_1(\lambda) = e'(0,-\lambda) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)e(0,-\lambda)$$

and

$$|S(\lambda)| = 1.$$

The scattering function $S(\lambda)$ is meromorphic in half plane $Im\lambda > 0$, with poles at the zeros of the function $E(\lambda)$. Moreover, the function $E(\lambda)$ is analytic in upper half plane. The function $E(\lambda)$ may have only a finite number of zeros in the half plane $Im\lambda > 0$.

We shall obtain the main equation that contributes to construct the potential $q(x)$ in the Eq. (1). To obtain the main equation, we substituting the relation (4) into the relation (6). Thus, the following results are valid:

Theorem 1. *For each fixed* $\neq 0$ *the kernel* $K(x,t)$ *satisfies the following equation:*

$$F(x+y) + K(x+y) + \int_x^\infty K(x,t)F(t+y)dt = 0, \quad x < y < \infty, \tag{8}$$

Proof. From [3, Theorem 3.1.] it is clear that the main equation can be constructed.

Thus, we have the following theorem.

Theorem 2. For each $x \geq 0$, the kernel $(K(x,t))$ to the solution (4) satisfy the main equation (8).

Uniqueness

Lemma 2. *Assume that the function $f_x(t)$ is summable on the half line for $t \geq x$*

$$f_x(t) + \int\limits_x^{\infty} f_x(u)F(u+t)du = 0, \tag{9}$$

and there is a solution for $f_x(t) \equiv 0$, $t \geq x$.

Proof. It can be easily seen from [3, Lemma 4.1], the function $f_x(t)$ be solution of the integral equation, where $K(x,t)$ satisfies the equation (8). Then, the homogeneous equation (9) has only trival solution i.e. $f_x(t) \equiv 0$ for $t \geq x$.

Theorem 3. *The scattering data of the boundary value problem* (1) – (3) *determine uniquely.*

Proof. Given scattering function $S(\lambda)$ for $\lambda \neq 0$ and the scattering data can be determined according to Eq. (8). By virtue of the function $F(x)$, the main equation is constructed and it sufficies to find only scattering data of the boundary value problem (1)–(3). Given the scattering data, we can use formulas as follows:

$$F_s(x) = \frac{1}{2\pi} \int\limits_x^{\infty} [1 - S(\lambda)]e^{i\lambda x} d\lambda,$$

$$F(x) = \sum_{j=1}^{n} f_j(x) + F_s(x),$$

and

$$p_j(x) = e^{-i\lambda_j x} f_j(x), \quad j = 1, 2, ..., n.$$

By Lemma 2, the main equation has a unique solution. Futhermore, we find the function $K(x,t)$. It follows from, appliying (5) we have

$$q(x) = -2\frac{d}{dx}K(x,x). \tag{10}$$

Thus, the potential $q(x)$ can uniquely be found from (10).The theorem is proved.

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