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МОДЕЛИРОВАНИЕ В ПРИКЛАДНОЙ ЭЛЕКТРОДИНАМИКЕ И ЭЛЕКТРОНИКЕ

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Сборник содержит научные работы, представленные на проведенном и организованном в рамках международной конференции Saratov Fall Meeting (SFM'2008) в сентябре 2008 г. в Саратовском государственном университете двенадцатом семинаре IEEE Saratov-Penza Chapter «Electromagnetics of Microwaves, Submillimeter and Optic Waves». Семинар организован Саратовской и Пензенской первичной ячейкой «Микроволновая теория и техника / Электронные приборы / Антенны и распространение радиоволн / Компоненты, упаковка и технология изготовления/Науки о плазме» (IEEE MTT/ED/AP/CPMT/PS Saratov–Penza Chapter), входящей в международную научную организацию Institute of Electrical and Electronic Engineers. Работы, выполненные саратовскими членами IEEE, учеными и молодыми специалистами, представляют оригинальные результаты по компьютерному моделированию в электродинамике, электронике СВЧ, акустоэлектронике, СВЧ технике, оптике.

Для работников научных учреждений, вузов, инженеров–разработчиков, может быть полезен аспирантам и студентам старших курсов факультетов: радиофизических, радиотехнических, электротехнических, энергетических.

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ВВЕДЕНИЕ

Компьютерное моделирование является основным мощнейшим инструментом исследования сложных систем и структур. Использование строгих компьютерных моделей позволяет адекватно производить их анализ, синтез или оптимизацию и зачастую вытесняет натурный эксперимент. Для большого числа рассматриваемых задач проведение натурного эксперимента чрезвычайно сложно или невозможно вовсе, поэтому развитие методов математического моделирования является чрезвычайно важным и актуальным.

В прикладной электронике и электродинамике, включая и оптику, использование строгих методов анализа и синтеза при моделировании означает применение алгоритмов на основе уравнений Максвелла и строгих решений уравнений движения. Важным элементом, влияющим на адекватность моделирования, служит корректное введение материальных уравнений и уравнений движения частиц, а также учет нелинейных свойств.

В последнее время все большее значение приобретают автоматизированные системы анализа и проектирования приборов, устройств и структур СВЧ, КВЧ и оптических диапазонов. Применение электродинамических методов происходит для всех частот используемых электромагнитных волн, включая и оптический диапазон, причем в оптике традиционные методы анализа вытесняются строгим электродинамическим рассмотрением. Наряду с традиционными частотными подходами к моделированию развиваются и пространственно-временные методы, что характеризует бурный прогресс прикладной нестационарной электродинамики и оптики. Другими актуальными современными направлениями, представленными в сборнике, являются моделирование наноструктур (включая квазипериодические структуры) и применение электродинамических методов к нелинейным задачам.

Девятый выпуск сборника продолжает серию публикаций трудов научных семинаров объединенной первичной ячейки (IEEE MTT/ED/AP/CPMT.PS Saratov–Penza Chapter) входящей в международную научную организацию Institute of Electrical and Electronic Engineers. Указанная ячейка создана летом 1995 г. в Саратове и Пензе. В сборник вошли труды, представленные в 2008 г. на очередном двенадцатом семинаре (Saratov–Penza Chapter Workshops), который являлся шестым семинаром данной первичной ячейки под названием «Workshop on Electromagnetics of microwaves, submillimeter and optic waves». С 2003 года семинары под указанным названием проводятся ежегодно в сентябре в рамках международной конференции «Saratov Fall Meeting» в Саратовском государственном университете.

INTRODUCTION

In recent time there was an increasing development of Computer Aid Design (CAD) methods and rigorous approaches for microwave electron devices, units and elements all over the world and in Russia particularly. These methods have been applied both for linear and nonlinear systems and structures in time and spectrum domains. There is growing interest in electromagnetic and optics to nanostructures and metamaterials.

The correct introduction of material and motion equations and using of strict electrodynamic models play important role in adequate numerical simulation of structures. Recently the nonstationary approach for electromagnetics and electronics stays more desirable and applicable. The nanostructures such as photonic crystals and metamaterials play the important role in modern science and cause the different methods of its simulation. These directions of modeling is also have mirrored in the present 9-th issue.

In 1995 on July 11 the IEEE Joint MTT/ED Chapter has been formed in Saratov and Penza under the sponsorship and help of Electron Devices and Microwave Theory and Techniques Societies (ED–S and MTT–S). Then it has been supported by Antennas and Propagation, Components, Packaging, and Manufacturing Technology and Nuclear and Plasma Science Societies (APS, CPMTS and NPSS), and now it is named as IEEE MTT/ED/AP/CPMT/NPSS Saratov–Penza Chapter included into the IEEE Russian Section.

This issue contains the papers presented at the 11-th IEEE MTT/ED/AP/CPMT Saratov–Penza Chapter Workshop named as “Electromagnetics of Microwaves, Submillimeter and Optic Waves” which has been held in conjunction with Saratov Fall Meeting at the Saratov State University in the September, 2008.

NONLINEAR SELF- FOCUSING OF BOUNDED BUNCHES OF MAGNETOSTATIC WAVES IN 2-D COUPLED FERROMAGNETIC FILMS

M.A. Malugina *(Member IEEE)*, E.N. Beginin *(Member IEEE)*, Yu.P. Sharaevskii *(Member IEEE)*

Saratov State University, Saratov, Russia
e-mail: maluginama@sgu.ru

Abstract - Results of research of effects of self-influence of magnetostatic waves in the 2D - structure consisting of two ferromagnetic films, on the basis of the numerical solution of system of two-dimensional nonlinear Schrödinger equations are presented. The basic features of these effects in comparison with similar in 2D-structure on the basis of one film are analyzed and possibility of use of the coupled structures for control of formation of 2D-nonlinear wave bunches and packages magnetostatic waves is considered.

1. Introduction

At present the scientific interest is attracted by researches of nonlinear effects of self-influence of waves in various mediums on the basis of the solution of system of coupled nonlinear Schrödinger equations (see, for example, [1-5]). It is necessary to notice, that such researches play the important role, first of all, in problems of nonlinear optics [1], in hydrodynamics, the physicist of plasma, in vacuum and solid-state electronics, and in other areas [2-5]. In particular, such problems play a basic role in a case when distribution of several types of normal waves is possible, for example, fast and slow waves of a spatial charge [4,5], a signal containing some spectral component [5], in case of need the account of interaction of the direct and reflected waves [3].

Such researches represent special interest in a case magnetostatic waves (MSW), propagating in ferromagnetic films since the nonlinear effects of self-influence caused by modulation instability of waves, here are shown at rather small power levels [6-8]. In particular, the system coupled Schrödinger equations was used in [8] for the description of MSW behavior at simultaneous excitation of two signals on different frequencies.

New possibilities of control of effects of self-influence on MSW arise at use in quality waveguide systems of multilayered ferromagnetic structures [9-12]. In such structures coupling leads to essential change of MSW dispersive characteristics, that also influences character of modulation instability of waves [12,13].

It is necessary to notice, that in the works specified above [9-12] on research of influence of coupling on effects of MSW self-influence in ferromagnetic structures the one-dimensional case was considered. To such problem there corresponds a situation when excitation of waves in boundless ferrite structure the antenna with a length is much more than MSW wave length. However certain interest is represented also by the researches directed on studying of the nonlinear phenomena in two-dimensional ferromagnetic structures in which distribution of 2D-wave bunches and packages takes place. The account of limited width of structure allows to analyse also the effects connected with reflexion of a wave from cross-section borders of a film. With reference to a single film results of such researches are adduced in [7,14] in which possibility of formation of 2D-channels of BVMSW, in particular, is shown.

The present work is devoted research of features of effects of self-influence MSW in 2D-structure consisting of two ferromagnetic films, on the basis of the numerical solution of system of two-dimensional nonlinear Schrödinger equations.

2. Theoretical model

The investigated structure consists of two ferromagnetic films, in thickness D and magnetization of the saturation M_0. The films are located in a plane (x, y) and divided by a dielectric layer in thickness d. Magnetic field $\overline{H}_0$ directed long an axis y, tangential to surfaces of films and orthogonal to the exciting antenna in length a what located between films. Along an axis y will propagate backward volume MSW (BVMSW) what has modulation instability in longitudinal and cross-section directions [6].

According to the model two-layer ferromagnetic structure described in [11,12], we will assume, that mutual influence of MSW in each of films is carried out through high-frequency magnetic fields. It allows in the equations of movement for a vector of magnetization and the equations of magnetostatic effective magnetic fields $\overline{H}_{1,2}$ in each film 1 or 2 to present as follows: $\overline{H}_{1,2} = \overline{H}^0_{1,2} + \overline{h}_{1,2} + K\overline{h}_{2,1}$, where $\overline{h}_{1,2}$ - variable high-frequency magnetic fields, $\overline{H}^0_{1,2}$ - constant components, K - coefficient of coupling between layers. Value K can be calculated on the basis of the linear theory (see, for example, [13]). Let's admit also, that nonlinearity of each film is defined only by size of variable magnetization of this film, i.e. the nonlinearity caused by change longitudinal components of the magnetic moment $M_{y_{1,2}}$, for each layer is set in the form of [11]: $M_{y_{1,2}} \approx M_0\left(1-|m_{1,2}|^2\right)$, where $m_{1,2}$ – variable magnetizations in films 1 and 2, accordingly, which values are defined by high-frequency magnetic fields both one, and other film. The important feature of the considered coupled ferromagnetic structures is separation of the dispersive curve concerning to MSW in a single film, on two, corresponding to two normal waves in structure – fast and slow - with wave numbers $\overline{k}_{f,s}$ [13], and group velocities $\overline{V}_{f,s} = \dfrac{\partial\omega}{\partial\overline{k}_{f,s}}$. On the basis of the made assumptions it is possible to receive system of the nonlinear equations for envelope amplitudes of high-frequency magnetizations of fast and slow waves $\varphi_{f,s}$, which for a considered case will look like:

$$\begin{aligned} &j\left(\frac{\partial}{\partial t} + V^{f,s}\frac{\partial}{\partial y}\right)\varphi_{f,s} + \frac{A_y^{f,s}}{2}\frac{\partial^2\varphi_{f,s}}{\partial y^2} + \frac{A_x^{f,s}}{2}\frac{\partial^2\varphi_{f,s}}{\partial x^2} - \\ &-\frac{B_{f,s}}{4}\left(|\varphi_f|^2 + |\varphi_s|^2\right)\varphi_{f,s} - \frac{B_s^f}{4}\left(\varphi_f\varphi_s^* + \varphi_s\varphi_f^*\right)\varphi_{s,f} + j\alpha_{f,s}\varphi_{f,s} = 0, \end{aligned} \tag{1}$$

where $A_x^{f,s} = \dfrac{\partial^2\omega}{\partial k_x^{f,s\,2}}$ - coefficients of dispersion spreading in a cross-section direction for fast and slow waves, $A_y^{f,s} = \dfrac{\partial^2\omega}{\partial k_y^{f,s\,2}}$ - coefficients dispersion in a longitudinal direction, $B_f = \left.\dfrac{\partial\omega}{\partial|\varphi_f|}\right|_{k_f}$, $B_s^f = \left.\dfrac{\partial\omega}{\partial|\varphi_f|}\right|_{k_s}$, $B_f^s = \left.\dfrac{\partial\omega}{\partial|\varphi_s|}\right|_{k_f}$, $B_s = \left.\dfrac{\partial\omega}{\partial|\varphi_s|}\right|_{k_s}$ - coefficients, characterizing nonlinearity, and depending on parameters of films and size of coupling K;

$\alpha_{f,s}$-parameter of dissipation. The index "*f*" concerns a fast wave with $\bar{k} = \bar{k}_f$ and an index "*s*" – to a slow wave with $\bar{k} = \bar{k}_s$. Unlike [11,12] system of the equations (1) it is written down for two-dimensional model, it was thus supposed, that longitudinal components $k_y^{f,s} >> k_x^{f,s}$ i.e. $V^{f,s} = \dfrac{\partial \omega}{\partial k_y^{f,s}}$. Amplitudes of normal waves $\varphi_{f,s}$ are associated with amplitudes of waves in each of the identical films $\varphi_{1,2}$ a following ratio [13]:

$$\varphi_{f,s} = \varphi_{1,2} \pm \varphi_{2,1}. \qquad (2)$$

The system (3) contains a number of differences from the similar NSE systems applied to the analysis of the nonlinear phenomena in other mediums (see, for example, [2-6,18,19]). The system concerns structure in which there are two waves extending with different group velocities that leads to occurrence of the cross nonlinear complex-conjugated members in the equations (3). Coefficients in (3) are defined not only MSW dispersion, but also depend on size of coupling and can vary in very wide limits [11,12]. Besides, in the equations dispersion spreading of envelope amplitude in a cross-section direction, and also damping in ferromagnetic films, that essentially for the analysis of MSW behavior, are considered.

3. Calculation results

On the basis of the received model researches of features of nonlinear wave processes and effects of self-influence were carried out in the 2D-coupled ferromagnetic structures.

Boundary conditions in a longitudinal direction along a magnetic field were set as: At $y = 0$ $\varphi_{f,s}(x,y,t) = \varphi_{0_{f,s}} Cos(\pi x/a)$, if $0 \le |x| \le a/2$ and $\varphi_{f,s}(x,y,t) = 0$, if $a/2 \le |x| \le L_x/2$; at $y = L_y$ $\varphi_{f,s}(x,y,t) = 0$, where L_y, L_x - lengths of structure in directions y and x. The case of excitation of system by a continuous signal of constant envelope amplitude fast and slow waves $\varphi_{0f,s}$ at y =0 was considered. Boundary conditions in a cross-section direction x were set in a kind of «a magnetic wall», i.e. at $x = \pm L_x/2$ $\varphi_{f,s}(x,y,t) = 0$. Initial conditions at $t = 0$ looked like $\varphi_{f,s}(x,0,0) = \varphi_{0f,a} Cos(\pi x/a)$.

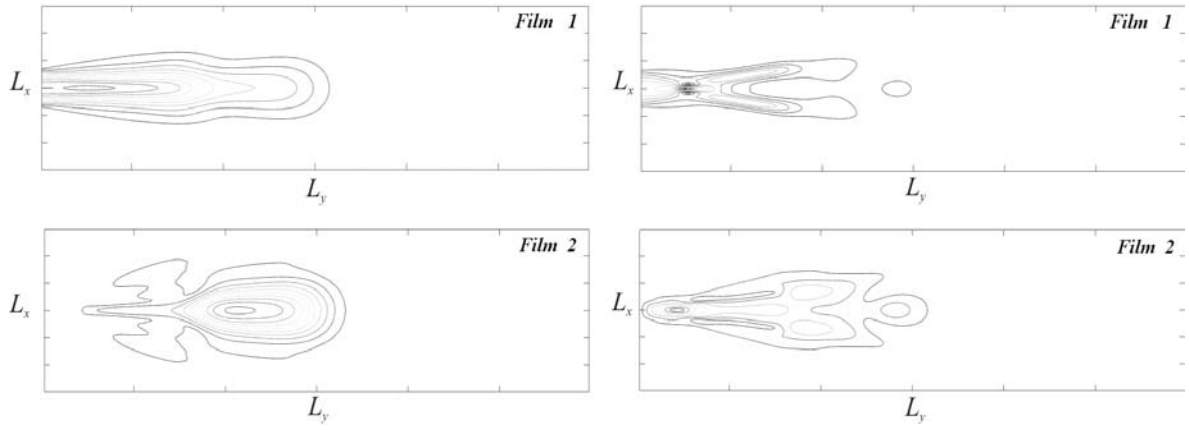
The analysis carried out for films by thickness D=7,2 μm with magnetization of saturation $4\pi M_0$=1750 Gs and length L_y=4 cm, placed in an external magnetic field $\overline{H}_0$=1391 Oe for which in [14] results of numerical research of nonlinear effects of self-influence are resulted at excitation dipole-exchange BVMSW on frequency f = 5,82 GHz with wave number k=63 cm^{-1} in single yttrium-iron garnet films. Coefficients in system (1) calculate off from a dispersive relationship for coupled dipole BVMSW [13], propagating in structure from two identical films with the parameters resulted above. Value of parameter of coupling K in such system lays in limits from ($K = 0$ at $d \to \infty$) to ($K = 1$ at for $d = 0$ a film of the doubled thickness). Calculations were spent at the fixed size of coupling K=0,4 that corresponds to a thickness of a dielectric interval $d = D$. Value of parameter of coupling got out so that thresholds of nonlinear effects on fast and slow waves differed weakly. Coefficients for fast and slow waves in system (1) at the chosen sizes of parameters of structure have following values: V^f=4,25·10^6 cm/s, V^s=2,75·10^6 cm/s,

A_y^f=3,41·10^3 cm^2/ s, A_x^f=2,14·10^5 cm^2/ s, A_y^s= 2,47·10^3 cm^2/s, A_x^s= 2,12·10^5 cm^2/s, $B_f^f = B_s^s = B_s^f = B_f^s$ =-7,3·10^9 s^{-1}, a case of absence of losses, i.e. $\alpha_{f,s}$=0 are presented.

If one normal wave is excited in structure the equations in (1) describe behavior of amplitude envelope amplitude in a single film and coincide with the similar equations in [6,7,14]. In this case, as shown in [7,14], at amplitude of an entrance signal above some threshold value, formation one or several stationary waveguide channels takes place.

If in structure it is excited two waves, fast and slow for the description of behavior envelope amplitude it is necessary to use the system (1). Excitation of two waves ($\varphi_{0f} \neq 0, \varphi_{0s} \neq 0$) is possible if to supply a signal only in one film ($\varphi_{01} = 2\varphi_0, \varphi_{02} = 0$ thus $\varphi_{0f} = \varphi_{0s} = \varphi_0$). The basic features of wave evolution in this case, unlike MSW behavior in a single film or the excitation of one wave considered above a case, will be caused, first of all, by existence of two normal waves with various velocities (V^f and V^s) in the structure.

At amplitude above some threshold value formation waveguide channels, both on fast, and on slow waves takes place. Because of different coefficients of a dispersion and nonlinearity of normal waves wave channels have various velocities and parameters. At the small amplitude of an entrance signal close to a threshold of self-focusing, the sizes of channels on fast and slow waves differ weakly (see the results presented on fig. 1) see. The basic role in wave evolution in this case is played by a difference of their group velocities.



a b

Fig. 1. Lines of equal level the amplitudes showing evolution of envelope amplitude in films 1 and 2 at excitation of two waves in structure at t=200 ns (a=0,25 cm, L_x=1 cm): (a) φ_0=0,005; (b) φ_0=0,06

If the signal originally supplies in a film 1 after some time t the difference in group velocities of fast and slow waves will lead to formation in a film 2 a perturbation of length $(V_f - V_s)t$ between forward fronts of fast and slow waves (see fig. 1a). Perturbation will be «lengthen out» and travel eventually by the structure end. After perturbation will travel by the structure end, the whole of power will be concentrated in a film 1. The similar picture will be, if a signal to supply only in a film 2.

At increase in amplitude of an entrance signal the channels formed on each of waves, have not only different velocities, but also the various characteristic sizes, and also there can be various a number of channels on each of waves. Joint distribution of fast and slow waves in this case will lead to formation of the stationary channel not only in a film 1, but also in a film 2 (in which the signal originally did not supply), as is shown in fig. 1b at φ_0=0.06.

At the further increase in amplitude of an entrance signal (fig. 2 see) energy of the basic bunch supplying in a film 1, divided already between three bunches extending in waveguide regime under some angle, it looks like from fig. 2 (and both in a film 1, and in a film 2 at that). Hence, it is possible to conclude, that existence of two waves in the coupled structure and various character of their nonlinear self-focusing leads to that the signal of the large amplitude supplied only in one of films, divided between films, and the signal with small amplitude – is not pumped over in other film.

Limitation of the cross-section sizes of a film leads to reflex ion of wave channels from boundaries. As a result, both in a single film, and in the coupled structure, the symmetric picture from minima and maxima envelope amplitude of a signal is created on width of films. However, in case the width of a film is close to the sizes of the exciting antenna in a single film takes place waveguide distribution of a signal with width of the wave front, coinciding with width of a film. In the coupled structure formation of channels on fast and slow waves, with various parameters and evolution in time, allows to observe following dynamics, which cannot be realized in a single film (see the results presented on fig. 3). If the sizes of a film coincide with the sizes of «a focal spot» (the area in a film plane, on some distance from the exciting antenna in which the width of a bunch is minimum), is observed stationary sequence of impulses in a film 2 (in which the signal originally did not supply) as is shown in fig. 3a, 3b. In a stationary state impulses remain practically immovable, and along a film 2 eventually only there is an increasing number of impulses.

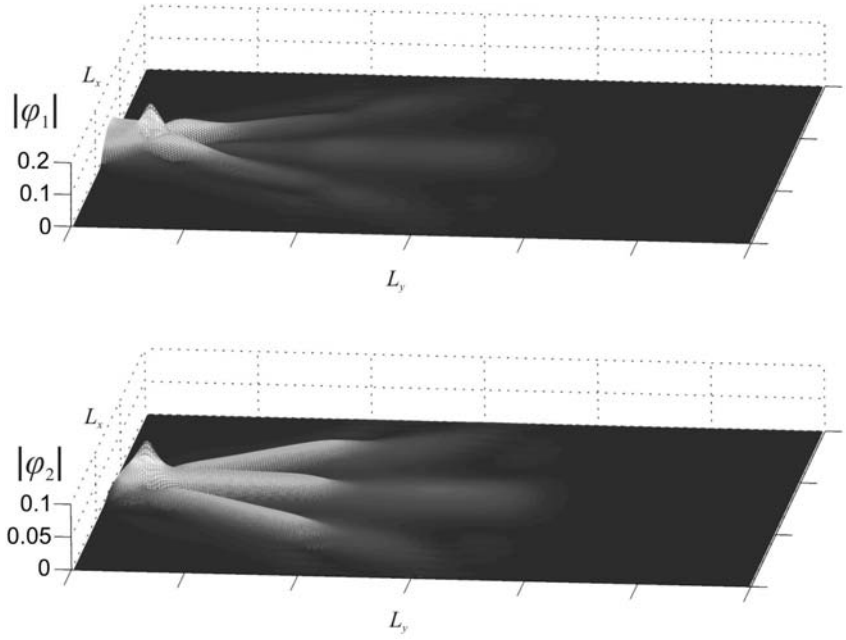
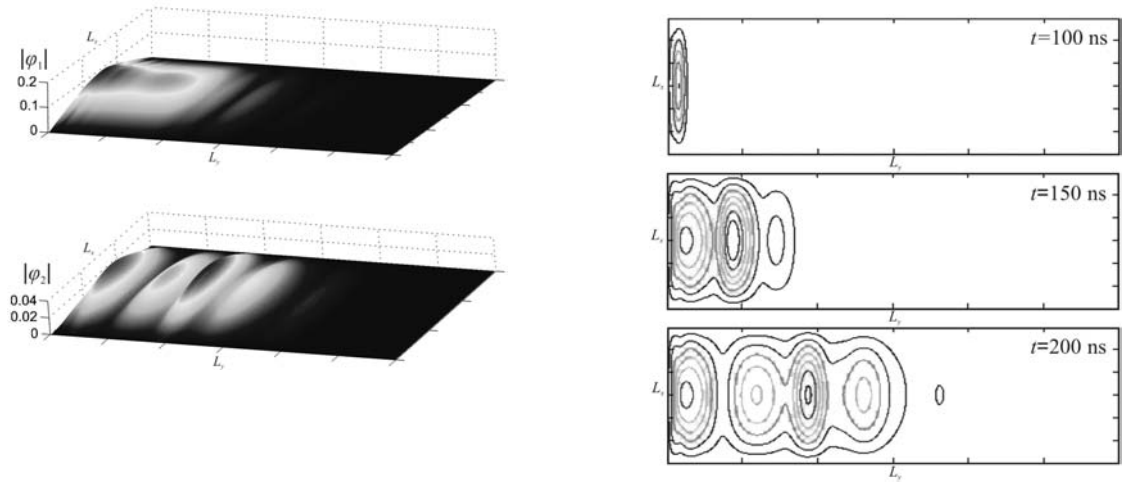


Fig. 2. Distribution of intensity of a wave bunch in films 1 ($|\varphi_1|$) and 2 ($|\varphi_2|$) in a plane (x, y) at t = 200 ns at excitation of two normal waves in structure (a =0,25 cm, L_x =4 cm, φ_0 =0,09)



a b

Fig. 3. Evolution of envelope amplitude in a plane (x, y) at $a = L_x$=0,25 cm: (a) in films 1 and 2; (b) lines of equal level of envelope amplitude in a film 2 ($|\varphi_2|$) during the various moments of time

4. Conclusion

On the basis of the numerical investigation the self-channeling caused by presence of coupling in ferromagnetic structure in comparison with similar effects in single 2D-structure are analyzed.

It's shown, that at excitation by a continuous signal various character of self-channeling for fast and slow waves leads to dependence of efficiency of swapping of a signal from one film in another from amplitude of a signal (there is «a nonlinear coupling»). In particular, for signals of the large amplitude there is an intensive swapping of a signal from one film in another, and for small amplitude – swapping is absent. Reflex ion from cross-section borders of films of fast and slow waves in system gives difficult interferential picture of behavior envelope amplitude in both films. Thus, if the width of the antenna is equal to width of a film and coincides with the sizes of «a focal spot», creation of stationary sequence of impulses along the second film is possible.

The results received in work concern a case of fixed value of coupling (K=Const). Change of size of coupling over a wide range, as shown in case of the analysis of one-dimensional similar structures, can lead to change of character of instability magnetostatic waves (see, for example, [12]), and it can be of interest as well with reference to 2D-structures. The last opens more ample opportunities of control of nonlinear effects in the 2D-coupled ferromagnetic film structures.

Acknowledgments

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H- AND LM- MODES OF RECTANGULAR AND SQUARE SECTION DIELECTRIC WAVEGUIDES

M.V. Davidovich, *Senior Member IEEE,* J.V. Stephuk

Saratov State University, 410012 Saratov, Russia
E-mail: DavidovichMV@info.sgu.ru

Abstract – The integral and integrodifferential equations and functionals in two-dimensional dielectric waveguide cross-section and the iteration algorithms of their solutions have been considered. The parameters of *H*-modes are determined numerically for square dielectric waveguide.

The rectangular dielectric waveguides (RDWs) have been considered and investigated for various practical applications in the series of works [1–30]. They have been used as probes, in the structures of filters and planar integrated circuits, as elements of resonators, for measurements of dielectric material properties, and for other purpose. Different approximate and rigorous approaches have been applied to analyze the RDW and its discontinuities. There are: the decomposition on the cylindrical harmonics [2,12,17], the Marcatili's approach [3], the transverse resonance method [13], the generalized telegraph equation method [9], the finite difference method [12,15,24], the mode matching method [6,16], the integral equation method [17,18,25,26] and the variation approach [5]. For numerical investigations of these approaches several numerical algorithms have been applied: the moment method, the Bubnov-Galerkin method, the collocations, the finite elements, the finite differences and some others. The dimensions of such approaches are very high when good accuracy is need. Therefore the precise determination of RDW mode parameters is actual [14]. In this paper we introduce the simple relations and effective algorithm for accurate determinations of eigenmodes (eigenwaves) and complex *H*-waves for RDW. The approach is based on two-dimensional hyper-singular integral equation and iteration procedure for its solution which allow one to determine parameters of modes with very high accuracy and small computing resources. Instead of usually used infinite-dimensional functional banach spaces we seek the solution in the two-dimensional Euclidean numeric space with analytical field presentation. The algorithm is easily generalized on other RDR modes and for nonuniform RDW in one direction.

Let consider the open RDW with constant permittivity $\varepsilon = 1 + \kappa$ in the region $|x| \le a$, $|y| \le b$, and equal to unit permittivity outside of them. We seek the solution in the form $\vec{E}(\vec{r}) = \vec{E}(\vec{r}_\perp)\exp(-j\gamma z)$. For the *H*- modes we have $E_z \equiv 0$, $\vec{E} = \vec{E}_\perp$. The transverse components in the dielectric region are presented evidently with respect of symmetry as:

$$E_x(\vec{r}_\perp) = \frac{\beta}{\chi}\begin{pmatrix}\cos(\alpha x)\\ \sin(\alpha x)\end{pmatrix}\begin{pmatrix}\cos(\beta y)\\ \sin(\beta y)\end{pmatrix}, \qquad E_y(\vec{r}_\perp) = \frac{\alpha}{\chi}\begin{pmatrix}\sin(\alpha x)\\ \cos(\alpha x)\end{pmatrix}\begin{pmatrix}\sin(\beta y)\\ \cos(\beta y)\end{pmatrix}, \tag{1,a}$$

$$E_x(\vec{r}_\perp) = \frac{\beta}{\chi}\begin{pmatrix}\cos(\alpha x)\\ \sin(\alpha x)\end{pmatrix}\begin{pmatrix}\sin(\beta y)\\ \cos(\beta y)\end{pmatrix}, \qquad E_y(\vec{r}_\perp) = -\frac{\alpha}{\chi}\begin{pmatrix}\sin(\alpha x)\\ \cos(\alpha x)\end{pmatrix}\begin{pmatrix}\cos(\beta y)\\ \sin(\beta y)\end{pmatrix}. \tag{1,b}$$

It should be taken separately either upper or lower significances in each of round brackets (first and second). Then they include all cases of solution symmetry or parity-oddness regarding to transverse coordinates. The electric field (1) is solenoidal: $\nabla_\perp \cdot \vec{E}_\perp = 0$. Here $\chi = \sqrt{k_0^2\varepsilon - \gamma^2} = \sqrt{\alpha^2 + \beta^2}$ is the transverse wave number in the dielectric. Moreover, the filed inside satisfies the wave (Helmholtz) equation inasmuch as we apply the condition

$$k_0^2\varepsilon = \alpha^2 + \beta^2 + \gamma^2. \tag{2}$$

It is convenient to classify the modes by component H_z parity-oddness properties relatively x and y coordinates using which one can express all other electric and magnetic components. Particularly, for electric filed (1) there are $E_x = -\left(j\omega\mu_0/\chi^2\right)\partial H_z/\partial y$, $E_y = \left(j\omega\mu_0/\chi^2\right)\partial H_z/\partial x$. It is need else to find two additional equations connecting three quantities (α,β,γ) in order to determine the propagation constant γ. We will use for this the hyper-singular integral equation (IE) for electric field in the cross-section S of arbitrary inhomogeneous dielectric waveguide (DW) [25,26]:

$$\vec{E}(\vec{r}_\perp) = \frac{-j}{4}\left[(\nabla_\perp - j\gamma\vec{z}_0)\otimes(\nabla_\perp - j\gamma\vec{z}_0) + k_0^2\right]\iint_S H_0^{(2)}\left(\chi_0|\vec{r}_\perp - \vec{r}_\perp'|\right)\kappa(\vec{r}_\perp')\vec{E}(\vec{r}_\perp')dx'dy' \quad . \qquad (3)$$

Here $\nabla_\perp = \vec{x}_0\partial/\partial x + \vec{y}_0\partial/\partial y$, the symbol $\otimes$ denotes the tensor (dyadic) multiplication, $H_0^{(2)}$ is the Hankel function of second kind and zero index, $\chi_0 = \sqrt{k_0^2 - \gamma^2}$ is transverse wave number in vacuum, the zero indexes denote the orthonormal coordinate vectors. Let $g(\vec{r}_\perp - \vec{r}_\perp') = -jH_0^{(2)}\left(\chi_0|\vec{r}_\perp - \vec{r}_\perp'|\right)/4$ denotes the two-dimensional Green's function (GF). The operator corresponding with equation (3) belongs to pseudo-differential ones [31]. In our case the equation has the form

$$\vec{E}_\perp(\vec{r}_\perp) = \kappa\left[\nabla_\perp\otimes\nabla_\perp + k_0^2\right]\iint_S g(\vec{r}_\perp - \vec{r}_\perp')\vec{E}_\perp(\vec{r}_\perp')dx'dy' \, . \qquad (4)$$

Let $\vec{F}_\perp(\vec{r}_\perp)$ is the continuously differentiable vector in the rectangular region S. We turn from the equation (4) to corresponded bilinear functional:

$$\Phi = \int_S \vec{F}_\perp(\vec{r}_\perp)\vec{E}_\perp(\vec{r}_\perp)dxdy - k_0^2\kappa\int_S\int_S \vec{F}_\perp(\vec{r}_\perp)g(\vec{r}_\perp - \vec{r}_\perp')\vec{E}_\perp(\vec{r}_\perp')dx'dy'dxdy - I = 0 \, . \qquad (5)$$

The equality to zero in the (5) is achieved on the exact solution. The integral I in (5) may be transformed as follows:

$$I = \kappa\int_S\int_S \vec{F}_\perp(\vec{r}_\perp)\nabla_\perp\otimes\nabla_\perp g(\vec{r}_\perp - \vec{r}_\perp')\vec{E}_\perp(\vec{r}_\perp')dx'dy'dxdy =$$

$$= \kappa\oint_L\int_S \vec{\nu}(\vec{r}_\perp)\cdot\vec{F}_\perp(\vec{r}_\perp)\nabla_\perp\cdot\left(g(\vec{r}_\perp - \vec{r}_\perp')\vec{E}_\perp(\vec{r}_\perp')\right)dS'dl - \kappa\int_S\int_S \nabla_\perp\cdot\vec{F}_\perp(\vec{r}_\perp)\nabla_\perp\cdot\left(g(\vec{r}_\perp - \vec{r}_\perp')\vec{E}_\perp(\vec{r}_\perp')\right)dS'dS. \qquad (6)$$

Here L is the bounding rectangular contour for rectangular cross-section, $dS \equiv d^2r \equiv dxdy$ is the elementary surface element, $\vec{\nu}$ is the unite normal to the contour vector in the plane x,y. It is convenient to take the function $\vec{F}_\perp$ so that her normal component is zero on the contour. Then the first integral in the (6) disappears. If one demands the condition $\nabla_\perp\cdot\vec{F}_\perp(\vec{r}_\perp) = 0$ then the second integral in the (6) disappears. On assumption of both conditions one has $I = 0$. The equality (5) gives the additional condition for γ determination. Fitting two vector-functions $\vec{F}_\perp(\vec{r}_\perp)$ one can get two necessary conditions. However, if the function $\vec{F}_\perp(\vec{r}_\perp)$ is chosen, then the $\vec{x}_0F_x(\vec{r}_\perp)$ and $\vec{y}_0F_y(\vec{r}_\perp)$ also are the same functions. Indeed, their normal components are equal to zero if it is justly for initial vector-function $\vec{F}$. Thus, choice of two-component vector-function defines two additional conditions.

It is expedient to choose the weight functions $\vec{F}$ in such way that they have the same symmetry as the desired solution. Then the functions with zero normal components on the contour (accurate within any multiplier) take the form

$$F_x(\vec{r}_\perp) = \begin{pmatrix}\cos(\alpha_m'x)\\ \sin(\alpha_m''x)\end{pmatrix}\begin{pmatrix}\beta_l'\cos(\beta_n'y)\\ \beta_l''\sin(\beta_n''y)\end{pmatrix}, \qquad F_y(\vec{r}_\perp) = \begin{pmatrix}\alpha_m'\sin(\alpha_m'x)\\ \alpha_m''\cos(\alpha_m''x)\end{pmatrix}\begin{pmatrix}\sin(\beta_l'y)\\ \cos(\beta_l''y)\end{pmatrix} \, , \qquad (7)$$

$$F_x(\vec{r}_\perp)=\begin{pmatrix}\cos(\alpha'_m x)\\ \sin(\alpha''_m x)\end{pmatrix}\begin{pmatrix}\beta''_l\sin(\beta''_l y)\\ \beta'_l\cos(\beta''_l y)\end{pmatrix},\qquad F_y(\vec{r}_\perp)=-\begin{pmatrix}\alpha'_m\sin(\alpha'_m x)\\ \alpha''_m\cos(\alpha''_m x)\end{pmatrix}\begin{pmatrix}\cos(\beta''_l y)\\ \sin(\beta'_l y)\end{pmatrix}. \tag{8}$$

Here $\alpha'_m=(m-1/2)\pi/a$, $\alpha''_m=m\pi/a$, $\beta'_l=l\pi/b$, $\beta''_l=(l-1/2)\pi/b$, $m,l=1,2,...$, and the functions (7)–(8) are solenoidal. They may be used as initial approaches to the solutions for the modes with appropriate indexes. It is possible to use also non solenoidal functions. The same, for example, are the functions $\vec{x}_0F_x(\vec{r}_\perp)$, $\vec{y}_0F_y(\vec{r}_\perp)$. At the same time the integral I may be defined for each function.

Substituting the relations (7)–(8) in to functional (5), we get the necessary number equations for α and β determination. Thus, using the two above named functions we have

$$\alpha=\left[k_0^2\kappa\int_S\int_S F_y(\vec{r}_\perp)g(\vec{r}_\perp-\vec{r}'_\perp)E_y(\vec{r}'_\perp)dx'dy'dxdy+I_y\right]/I_{0y}\quad, \tag{9}$$

$$\beta=\left[k_0^2\kappa\int_S\int_S F_x(\vec{r}_\perp)g(\vec{r}_\perp-\vec{r}'_\perp)E_x(\vec{r}'_\perp)dx'dy'dxdy+I_x\right]/I_{0x}\quad. \tag{10}$$

Here I_x and I_y are the significances of integral (6), $I_x=-I_y$, and βI_{0x} and αI_{0y} are the significances of the first integral in (5) for these two function. As example, for the upper functions in the (1,a) there is

$$I_{0x}=\int_{-a-b}^{a\ b}\cos(\alpha x)\cos(\alpha'_m x)\cos(\beta y)\cos(\beta'_l y)dxdy=$$

$$=\left[\frac{\sin((\alpha+\alpha'_m)a)}{\alpha+\alpha'_m}+\frac{\sin((\alpha-\alpha'_m)a)}{\alpha-\alpha'_m}\right]\left[\frac{\sin((\beta+\beta'_l)b)}{\beta+\beta'_l}+\frac{\sin((\beta-\beta'_l)b)}{\beta-\beta'_l}\right],$$

$$I_{0y}=\int_{-a-b}^{a\ b}\sin(\alpha x)\sin(\alpha'_m x)\sin(\beta y)\sin(\beta'_l y)dxdy=$$

$$=\left[\frac{\sin((\alpha-\alpha'_m)a)}{\alpha-\alpha'_m}-\frac{\sin((\alpha+\alpha'_m)a)}{\alpha+\alpha'_m}\right]\left[\frac{\sin((\beta-\beta'_l)b)}{\beta-\beta'_l}-\frac{\sin((\beta+\beta'_l)b)}{\beta+\beta'_l}\right].$$

Correspondingly, the integrals in the numerators of (9)–(10) may be introduced as:

$$I_x=-I_y=-\kappa\int_S\int_S\frac{\partial}{\partial x}F_x(\vec{r}_\perp)\left[E_x(\vec{r}'_\perp)\frac{\partial}{\partial x}g(\vec{r}_\perp-\vec{r}'_\perp)+E_y(\vec{r}'_\perp)\frac{\partial}{\partial y}g(\vec{r}_\perp-\vec{r}'_\perp)\right]dS'dS=$$

$$=\kappa\int_S\int_S\frac{\partial}{\partial yx}F_y(\vec{r}_\perp)\left[E_x(\vec{r}'_\perp)\frac{\partial}{\partial x}g(\vec{r}_\perp-\vec{r}'_\perp)+E_y(\vec{r}'_\perp)\frac{\partial}{\partial y}g(\vec{r}_\perp-\vec{r}'_\perp)\right]dS'dS.$$

Instead of equations (9)–(10) one can use similar relations for the functions (7)–(8) under the different indexes, and also for other functions. Such equations with the relation (2) allow to determine the eigenmodes with the parameters α_n, β_n, γ_n for each k_0, where $n=0,1,2,....$ denotes their numeration. It is appropriate to introduce this numeration for each group of modes having the adjusted symmetry (for example, with even component E_x relative to both coordinates). It must notice that α_n and β_n depend from frequency and do not coincide with the significances $m\pi/(2a)$ and $l\pi/(2b)$, where m and l are the integers (the numbers of field variations with the coordinates). Therefore the classification like H_{ml} is not quite successful. If the permittivity ε is real then there are the slow surface eigenmodes with $\gamma_n>k_0$. In this case the transversal wavenumber χ_0 is imaginary, and GF is presented by MakDonald's function K_0 and is real. There is finite number of such functions for any given frequency. In the opposite case the GF is complex, and the problem

is not self-conjugate. Complex in this case will be the modes and their parameters. All modes are complex in presence of dielectric loss i.e. for the case $\varepsilon = \varepsilon' - j\varepsilon''$.

The mode parameters are entering nonlinearly into the equations. To determine these, it is proper to use the iteration methods and algorithms. If the principle of contraction is fulfilled then iteration approaches converge to appointed value from any initial approximation in this region. Let such solutions α_0, β_0, γ_0 are determined. The corresponding transverse function is $\vec{E}_0(x,y)$. The test functions for the second mode determination must be orthogonal to $\vec{E}_0$. It is obvious, this is

$$\vec{\tilde{E}}(x,y) = \vec{E}_\perp(x,y) - \vec{E}_0(x,y)\left\langle \vec{E}_\perp(x,y), \vec{E}_0(x,y)\right\rangle / \left\|\vec{E}_0(x,y)\right\|^2,$$

where the Dirac's brackets denote the innerproduct in the form of the integral over *S* from product of functions, $\|\ \|$ denotes the norm, and $\vec{E}_\perp$ is determined from (1).

Other based on the equation (4) approach follows from the integration by parts and construction of quadratic functional. Let write the (4) in the form

$$\kappa^{-1}E_x(\vec{r}_\perp) = k_0^2\int_S g(\vec{r}_\perp - \vec{r}_\perp')E_x(\vec{r}_\perp')dx'dy' + \frac{\partial}{\partial x}\int_S\left[E_x(\vec{r}_\perp')\frac{\partial g(\vec{r}_\perp - \vec{r}_\perp')}{\partial x} + E_y(\vec{r}_\perp')\frac{\partial g(\vec{r}_\perp - \vec{r}_\perp')}{\partial y}\right]dx'dy',$$

$$\kappa^{-1}E_y(\vec{r}_\perp) = k_0^2\int_S g(\vec{r}_\perp - \vec{r}_\perp')E_y(\vec{r}_\perp')dx'dy' + \frac{\partial}{\partial y}\int_S\left[E_x(\vec{r}_\perp')\frac{\partial g(\vec{r}_\perp - \vec{r}_\perp')}{\partial x} + E_y(\vec{r}_\perp')\frac{\partial g(\vec{r}_\perp - \vec{r}_\perp')}{\partial y}\right]dx'dy'.$$

Then the right integrals are integrating by parts. For example:

$$\int_S E_x(\vec{r}_\perp')\frac{\partial^2 g(\vec{r}_\perp - \vec{r}_\perp')}{\partial x^2}dx'dy' = \int_S E_x(\vec{r}_\perp')\frac{\partial^2 g(\vec{r}_\perp - \vec{r}_\perp')}{\partial x'^2}dx'dy' = \int_S g(\vec{r}_\perp - \vec{r}_\perp')\frac{\partial^2 E_x(\vec{r}_\perp')}{\partial x'^2}dx'dy' -$$

$$-\int_{-b}^{b}\left[g_x(x-a,y-y')E_x(a,y') - g_x(x+a,y-y')E_x(-a,y')\right]dy' -$$

$$-\int_{-b}^{b}\left[g(x-a,y-y')\frac{\partial E_x(a,y')}{\partial x'} - g(x+a,y-y')\frac{\partial E_x(-a,y')}{\partial x'}\right]dy',$$

$$\int_S E_y(\vec{r}_\perp')\frac{\partial^2 g(\vec{r}_\perp - \vec{r}_\perp')}{\partial x\partial y}dx'dy' = \int_S E_y(\vec{r}_\perp')\frac{\partial^2 g(\vec{r}_\perp - \vec{r}_\perp')}{\partial x'\partial y'}dx'dy' = \int_S g(\vec{r}_\perp - \vec{r}_\perp')\frac{\partial^2 E_y(\vec{r}_\perp')}{\partial x'\partial y'}dx'dy' -$$

$$-\int_{-a}^{a}\left[g_x(x-x',y-b)E_y(x',b) - g_x(x-x',y+b)E_y(x',-b)\right]dx' -$$

$$-\int_{-b}^{b}\left[g(x-a,y-y')\frac{\partial E_y(a,y')}{\partial y'} - g(x+a,y-y')\frac{\partial E_y(-a,y')}{\partial y'}\right]dy' = \int_S g(\vec{r}_\perp - \vec{r}_\perp')\frac{\partial^2 E_y(\vec{r}_\perp')}{\partial x'\partial y'}dx'dy' -$$

$$-\int_{-b}^{b}\left[g_y(x-a,y-y')E_y(a,y') - g_y(x+a,y-y')E_y(-a,y')\right]dy' -$$

$$-\int_{-a}^{a}\left[g(x-x',y-b)\frac{\partial E_y(x',b)}{\partial x'} - g(x-x',y+b)\frac{\partial E_y(x',-b)}{\partial x'}\right]dx'.$$

The remaining integrals result by index replacement $x \leftrightarrow y$. by virtue of electric field solenoidality the first equation is transformed thus:

$$\kappa^{-1}E_x(\vec{r}_\perp) = k_0^2\int_S g(\vec{r}_\perp - \vec{r}_\perp')E_x(\vec{r}_\perp')dx'dy' -$$

$$-\int_{-b}^{b}\left[g_x(x-a,y-y')E_x(a,y') - g_x(x+a,y-y')E_x(-a,y')\right]dy' -$$

$$-\int_{-a}^{a}\left[g_x(x-x',y-b)E_y(x',b)-g_x(x-x',y+b)E_y(x',-b)\right]dx'.$$

The second equation follows by replacement $x \leftrightarrow y$. Obviously, it is sufficient to conside two cases from four cases determined by (1). Let, for example, consider the first: $E_x=(\beta/\chi)\cos(\alpha x)\cos(\beta y)$, $E_y=(\alpha/\chi)\sin(\alpha x)\sin(\beta y)$. Multiplying the first equation on E_x and integrating over the S, one can get the equation for α/β:

$$\kappa^{-1}I_{0x}-k_0^2\int_0^a\int_0^b\int_0^a\int_0^b\cos(\alpha x)\cos(\beta y)G^{ee}(x,y\mid x',y')\cos(\alpha x')\cos(\beta y')dx'dy'dxdy+$$
$$+\int_0^a\int_0^b\int_0^b\cos(\alpha x)\cos(\beta y)G_{xx}^{ee}(x,y\mid y')\cos(\alpha a)\cos(\beta y')dy'dxdy=$$
$$=\frac{\alpha}{\beta}\int_0^a\int_0^b\int_0^a\cos(\alpha x)\cos(\beta y)G_{xy}^{ee}(x,y\mid x')\sin(\alpha x')\sin(\beta b)dx'dxdy.$$

Here the integral I_{0x} is defined for $\alpha'=\alpha,\beta'=\beta$ and the following kernels are introduced:

$$G^{ee}(x,y\mid x',y')=\left[g(x-x',y-y')+g(x+x',y-y')+g(x-x',y+y')+g(x+x',y+y')\right],$$
$$G_{xx}^{ee}(x,y\mid y')=\left[g_x(x-a,y-y')+g_x(x-a,y+y')-g_x(x+a,y-y')-g_x(x+a,y+y')\right],$$
$$G_{xy}^{ee}(x,y\mid x')=\left[g_x(x-x',y-b)+g_x(x-x',y+b)-g_x(x+x',y-b)-g_x(x+x',y+b)\right].$$

The second equation for β/α is derived in a similar way. Using the vector form of above equations, we have the quadratic functional for k_0^2:

$$k_0^2=\frac{\kappa^{-1}\int_S\vec{E}_\perp^2(\vec{r}_\perp)dS+\oint_L\oint_L\vec{v}(\vec{r}_\perp)\cdot\vec{E}_\perp(\vec{r}_\perp)g(\vec{r}_\perp-\vec{r}_\perp')\vec{v}(\vec{r}_\perp')\cdot\vec{E}_\perp(\vec{r}_\perp')dldl'}{\int_S\int_S g(\vec{r}_\perp-\vec{r}_\perp')\vec{E}_\perp(\vec{r}_\perp)\cdot\vec{E}_\perp(\vec{r}_\perp')dSdS'}. \qquad (11)$$

Here the first integral in the numerator is equal $\kappa^{-1}\left(\beta^2I_{0x}+\alpha^2I_{0y}\right)/\chi^2$, and the integral in the denominator has the form

$$\int_0^a\int_0^b\int_0^a\int_0^b\left[E_x(x,y)G_x(x,y\mid x',y')E_x(x',y')+E_y(x,y)G_y(x,y\mid x',y')E_y(x',y')\right]dx'dy'dxdy.$$

By its symmetry the double loop integral in the numerator is converted to the sum of two-dimensional integrals with limits on integrals $(0,a)$ and $(0,b)$. As it is easily seen, it is equal to zero. The introduced kernels are depended on mode type. Thus, if E_x is even relatively both arguments then $G_x\equiv G^{ee}$ (symbol "e" denotes the parity). The functional (11) depends nonlinearly depends on k_0^2 and it may be used to determine wave number dependence on γ. Particularly, setting the γ for the case of square waveguide with symmetrical mode $\alpha=\beta$ and zero-order approximation for k_0, one can get the relation $\alpha=\sqrt{\left(k_0^2\varepsilon-\gamma^2\right)/2}$ which allows to determine the next order approximation for k_0 from (11), i.e. to construct the dispersion low $\gamma(k_0)$.

The dispersion for quadratic DW with permittivity $\varepsilon=4$ is presented at Fig. 1, 2. The calculations have been performed by getting the stationary values of functional (11), which have been obtained by successive approximations method (SAM). The SAM rate of convergence near the critical frequencies is strongly reduced and the region of convergence is converged. Therefore it is impossible to approach closely to critical frequencies.

In the regions near the critical frequencies there are the mode jumps to the neighboring highest mode branches, i.e. each γ corresponds with the highest k_0 obtained by SAM. Sometimes for two first modes near the second critical frequency it is possible to find two k_0 values. Note, that GF turn to infinity at the critical frequency. The integrals

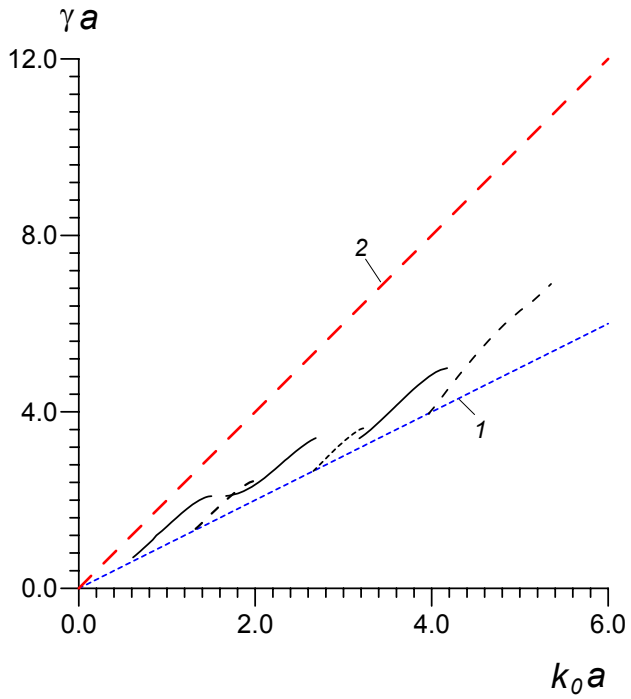


Fig. 1. Dispersion in the quadratic dielectric waveguide: solid curves – modes $H^{eo}_{n+\delta,n+\delta}$, dashed curves – modes $H^{ee}_{n+\delta,n+\delta}$, dashed straight line – $\gamma = k_0$ and $\gamma = k_0\sqrt{\varepsilon}$

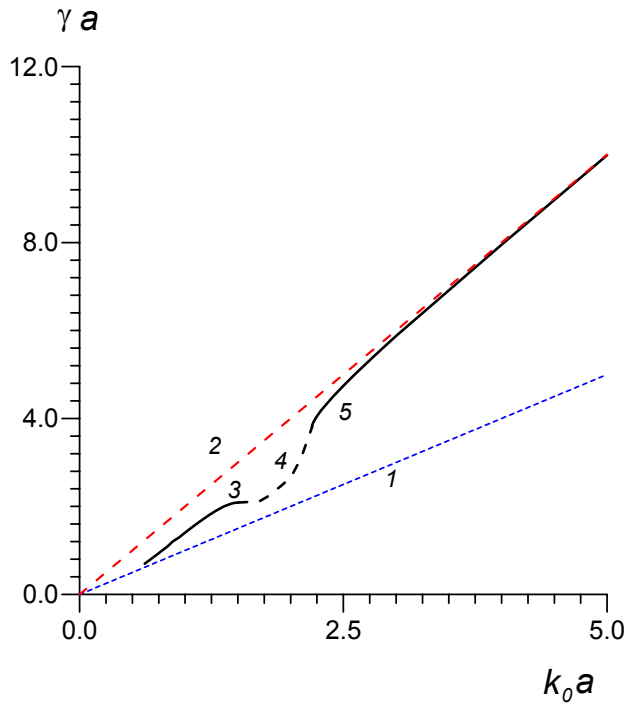


Fig. 2. Dispersion of the mode $H^{eo}_{\delta\delta}$ of square section dielectric waveguide with c $\varepsilon = 4.0$: the curves 3, 5 have been obtained by iteration method, the curves 1 and 2 correspond to lines $\gamma = k_0$ and $\gamma = k_0\sqrt{\varepsilon}$, the curve 4 has been obtained by bisection method

in (11) have been calculated by mean-value method and decomposition of intervals $(0,a)$ and $(0,b)$ on N parts with taking into account the symmetry.

The singularity has been taken into account by extraction of corresponding elementary regions (for which $\vec{r}_\perp = \vec{r}'_\perp$) and by the analytical integration over the equivalent circular regions with equal square a^2/N^2 (for $a = b$). The results of lower mode $H^{eo}_{\delta\delta}$ calculation are produced in the table for two values N. We have used such classification: index "e" on the fist place denotes the component H_z parity with regard to x, and on the second place – relative to y. Correspondingly index "o" denotes the oddness. The index δ is not integer and depended from frequency. It designates the number of field variation along the coordinates, i.e. is is determined from the equation $\alpha = (\delta + n)\pi/(2a)$. For quadratic DW the modes are doubly degenerate. After the passage from dispersion branches the index δ gets the jump. Therefore it is convenient to introduce such mode classification: $H^{\mu\nu}_{(n+\delta)(n+\delta)}$. Here μ,ν have the values e,o, and mode number n – values 0,1,2,... Then the main mode has $n = 0$.

The main mode $H^{eo}_{\delta\delta}$ dispersion is presented in the Fig. 2. The first part of curve corresponds with Fig. 1. The second part of curve has been obtained by iteration of (9). In the region of second critical frequency and lower this iterations do not converge. And the convergence to main mode takes place for the functional (11) in the region lower the second crucial frequency. Here there is the analogy with dielectric resonators for which the iterations converge to the mode with lowest resonant frequency.

The mode $H^{eo}_{\delta\delta}$ parameters for quadratic DW under the two numbers N of integration points

γa	$k_0 a$		α		δ	
	N=3	N=5	N=3	N=5	N=3	N=5
0.70	0.61597	0.61642	0.71682	0.71600	0.45634	0.45683
0.75	0.64525	0.64606	0.74307	0.74489	0.47305	0.47365
0.80	0.67424	0.67485	0.76756	0.76759	0.48866	0.48935
0.90	0.72989	0.73066	0.81269	0.81408	0.51738	0.51826
1.00	0.78405	0.78461	0.85409	0.85512	0.54373	0.54438
1.10	0.83742	0.83847	0.89051	0.89503	0.56853	0.56979
1.20	0.89049	0.89168	0.93056	0.93485	0.59241	0.59327
1.30	0.94364	0.94499	0.96742	0.97004	0.57735	0.61755

Let consider the algorithm generalization for inhomogeneous in one y-direction RDW, i.e. let consider that permittivity is the function of this coordinate: $\varepsilon = \varepsilon(y)$. In the case of RDW symmetry it is usually even function. Let use the piecewise constant its approximation. It means that there are the coordinate massive y_l, for which $y_0 = 0$, $y_N = b/2$, and in each layer the permittivity has the constant value ε_l, $l = 1,2,...,N$. We determine the field by the functions (1) with parameters α, β_l, γ. Inasmuch as the component E_x continuous, and the component E_y has the jump at the division boundaries, there are only two independent parameters from $N+1$ parameters α, β_l. Let take such α and β_1, and the remaining we will expressed through β_1. For example, for above considered case we have

$$\beta_l \cos(\beta_l y_l) = \beta_{l+1} \cos(\beta_{l+1} y_l), \qquad \varepsilon_l \sin(\beta_l y_l) = \varepsilon_{l+1} \sin(\beta_{l+1} y_l), \tag{12}$$

and $\beta_{l+1} = \beta_l \cos(\beta_l y_l)/\sqrt{1-(\varepsilon_l/\varepsilon_{l+1})^2 \sin^2(\beta_l y_l)}$, $l = 1,2,...,N-1$. If the function $\varepsilon(y)$ is neither even nor odd then no y symmetry, therefore the dependence must be taken as $A_l \sin(\beta_l y) + B_l \cos(\beta_l y)$. In this case the layers may be unsymmetrical placed in the RDW region $0 \le y \le a$. For N layers we have the $3N+1$ unknown magnitudes, the $2(N-1)$ conditions of kind (12) and the N conditions of kind (2). One more condition we impose taking $A_1 = 1$ by force of problem homogeneity. Thus, again we have the two unknown parameters. It is easier to express the amplitudes in the layers via the amplitudes in the first layer using the transfer matrix. Note, that the odd function $\varepsilon(y)$ for $y < 0$ means symmetric negative permittivity values, which hypothetical (only at one frequency) may correspond to plasma RDW with y-nonuniform (nonequilibrium) charge carriers distribution.

At last, let consider the extension for the *E*- modes and for hybrid modes *HE* and *EH*. There are the three *E*-filed components in all these cases. For *E*- mode more proper to use the formulating relatively the two magnetic field components. In this case the singular integrodifferential equation is

$$\vec{H}(\vec{r}_\perp) = (\nabla_\perp - j\gamma\vec{z}_0) \times \int_S g(\vec{r}_\perp - \vec{r}'_\perp)\hat{\kappa}(\vec{r}'_\perp)\hat{\varepsilon}^{-1}(\vec{r}'_\perp)(\nabla_\perp - j\gamma\vec{z}_0) \times \vec{H}(\vec{r}'_\perp)d^2r' \quad . \tag{13}$$

It is offered in the general form for the inhomogeneous anisotropic dielectric and means the availability of tri-component magnetic field. The *E*- modes in the uniform RDW are described by the equation

$$\vec{H}_\perp(\vec{r}_\perp) = \frac{1-\varepsilon}{\varepsilon}\left[\nabla_\perp \times \int_S g(\vec{r}_\perp - \vec{r}'_\perp)\nabla'_\perp \times \vec{H}_\perp(\vec{r}'_\perp)d^2r' + \gamma^2 \int_S g(\vec{r}_\perp - \vec{r}'_\perp)\vec{H}_\perp(\vec{r}'_\perp)d^2r'\right] \quad . \tag{14}$$

The test-functions for magnetic field are the same as (1). The integration in (14) with corresponding weight function leads to the functional with weakly singular (logarithmically singular) kernel. If one uses the electric field formulation then it is necessary to introduce the additional term into (4) and also the additional equation for longitudinal component:

$$\vec{E}_\perp(\vec{r}_\perp) = \kappa\left\{\left[\nabla_\perp \otimes \nabla_\perp + k_0^2\right]\iint_S g(\vec{r}_\perp - \vec{r}'_\perp)\vec{E}_\perp(\vec{r}'_\perp)d^2r' - j\gamma\nabla_\perp \int_S g(\vec{r}_\perp - \vec{r}'_\perp)E_z(\vec{r}'_\perp)d^2r'\right\}, \tag{15}$$

$$E_z(\vec{r}_\perp) = \kappa\left\{-j\gamma\nabla_\perp \cdot \int_S g(\vec{r}_\perp - \vec{r}'_\perp)\vec{E}_\perp(\vec{r}'_\perp)d^2r' + \chi_0 \int_S g(\vec{r}_\perp - \vec{r}'_\perp)E_z(\vec{r}'_\perp)d^2r'\right\}. \tag{16}$$

In the case of *E*- mode with even component E_z relatively *x* and *y* we have

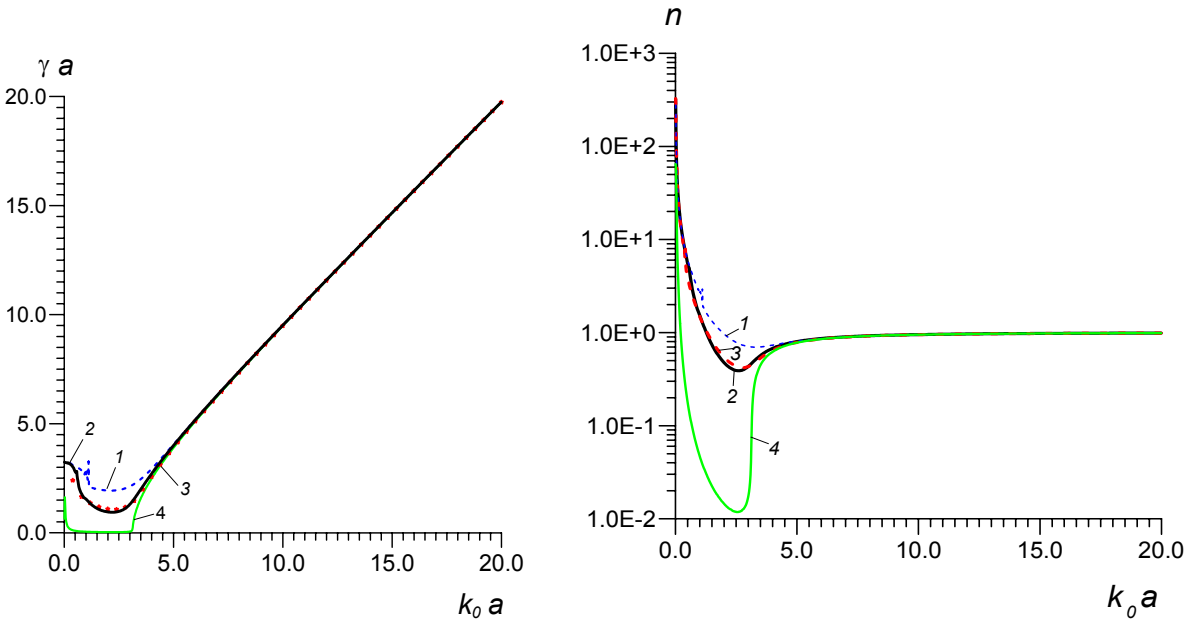
$$E_x(\vec{r}) = -(\alpha\gamma/\chi^2)\sin(\alpha x)\cos(\beta y)\exp(-j\gamma z),$$
$$E_y(\vec{r}) = -(\beta\gamma/\chi^2)\cos(\alpha x)\sin(\beta y)\exp(-j\gamma z),$$
$$E_z(\vec{r}) = j\cos(\alpha x)\cos(\beta y)\exp(-j\gamma z).$$

In general case for such hybrid mode one must use

$$E_x(\vec{r}) = (\gamma/\chi)^2[A\beta - B\alpha]\sin(\alpha x)\cos(\beta y)\exp(-j\gamma z),$$
$$E_y(\vec{r}) = -(\gamma/\chi)^2[A\alpha + B\beta]\cos(\alpha x)\sin(\beta y)\exp(-j\gamma z),$$
$$E_z(\vec{r}) = jB\cos(\alpha x)\cos(\beta y)\exp(-j\gamma z).$$

According to homogeneity of the problem we can arbitrary set one of amplitude parameters, putting for example $B = 1$. Consequently, there are three scalar equations (15)–(16) for tree parameters determination. The modes with different symmetries have similarly forms. Often the other classifications are used: E^x_{mn} or E^y_{mn} in [3], or even $HE_\perp$ - mode and odd $HE_\parallel$ - mode according to Schlosser [1] (when $a > b$). Concerning the modes with

$E_y = 0$ or $E_x = 0$ may be considerable. They may be treated as *H*- modes concerning *x* or *y* axes, or as modes *LE*. The indexes *m* and *n* here are also integer only approximately. Evident, the test functions for such even mode are $E_x = \gamma\cos(\alpha x)\cos(\beta y)$, $E_z = j\alpha\sin(\alpha x)\cos(\beta y)$.



a b

Fig. 3. Dispersion $\gamma = k_z'$ for LM_2- mode (a) and deceleration (b) in the plane-parallel waveguide at the following parameters of infinite coats: $\varepsilon = 3.0$ (the curve 1); $\varepsilon = 12.0$ (2); $\varepsilon = 12.0 - j12.0$ (3 – daggers); $\varepsilon = 1.0 - j10^4$ (4)

The plane-parallel waveguide structures have been also considered in this paper. In this case the rigorous dispersion equation [33]

$$Z^e + jZ_w^e \tan(k_x a/2) = 0 \qquad (17)$$

had been sole using the iteration approach. Here the dimension *b* is infinite, Z_w^e is the wave core impedance and Z^e is the covering impedance transformed to the core at $x = a$. The multilayered and homogeneous finite and infinite coating has been considered. The results of the equation (17) solutions are presented in the fig. 3-5 for different electro-physical parameters of coating and hollow core. The iterations have been performed using the step by step impedances, k_x and k_z calculations from high frequencies with little step. The initial values for k_x have been taken as the values at previous frequency value. The fig. 1–2 present the results for mode with index *m*=1 for real and complex permittivity values. The retardation is less then unity and weakly changing. It sharply falls down and reaches the its minimum and the loss increases at some frequency. At very low frequencies the wave is slow and leaky with great attenuation. The slowness in this case is caused by wave penetration into coating which refractive index is greater than unity and the reflection coefficient for near $\pi/2$ glancing angle is not small. The fig. 3 demonstrates the deceleration for first four modes when the coating is the copper with $\varepsilon = 10 - j\sigma_0/(\varepsilon_0\omega)$

and the conductivity $\sigma_0 = 5.8 \cdot 10^7$ S/m. For the quasi-T- mode in the wide frequency band the deceleration n is practically equal to unity and increases at very low frequencies. In the region sufficiently low than cutoff we have $n << 1$, but there is the sharp increase up to values $n >> 1$ in the region of superlow frequencies. The cutoff is defined by the condition $k_x a = 2m\pi$ and corresponds to the condition $\sigma_0 = \infty$. The multilayered coatings including the dielectric and semiconductor layers have been investigated also. For the semiconductor plasma there is three cases. The first corresponds to low frequencies when ε' is negative. The second one corresponds to the real part coating permittivity values $0 < \varepsilon' < 1$, and the third when $\varepsilon' > 1$ corresponds to the hollow center channel waveguide which is realized at high frequencies.

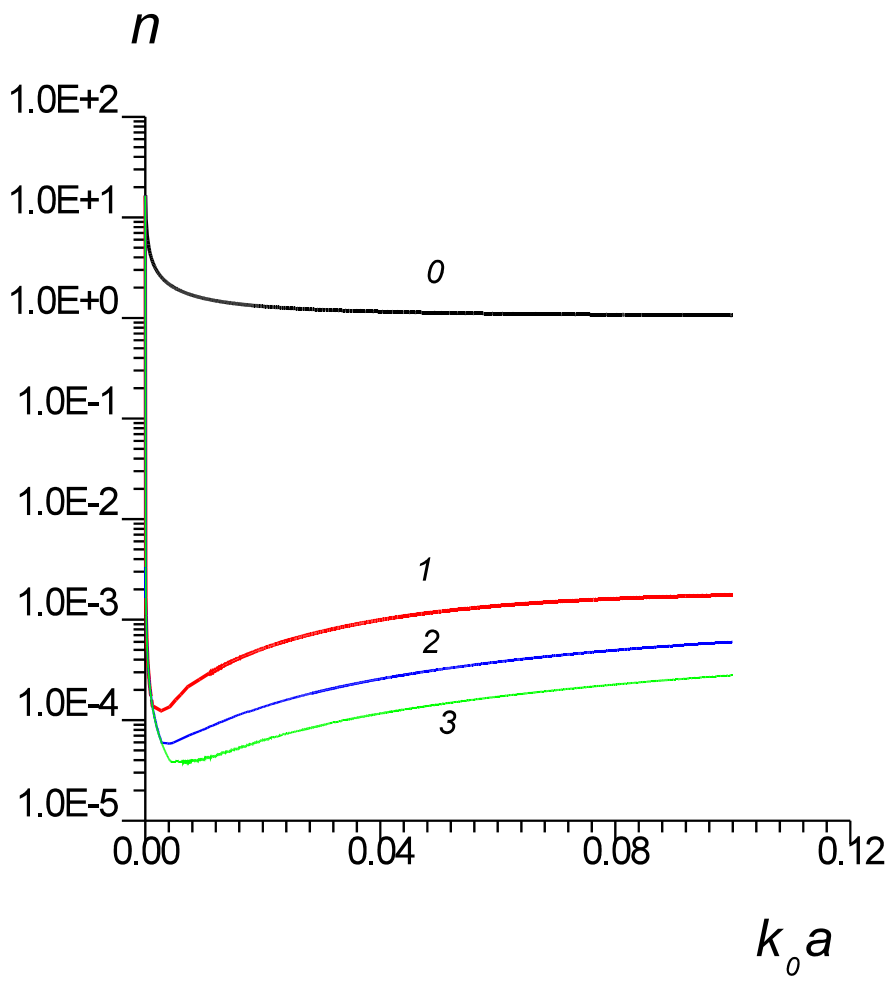


Fig. 4. Deceleration of even LM_{2m}- modes ($m=0,1,2,3$) for small in the case $\varepsilon = 10 - j\sigma_0 / (\varepsilon_0 \omega)$, $\sigma_0 = 5.8 \cdot 10^7$ S/m (copper)

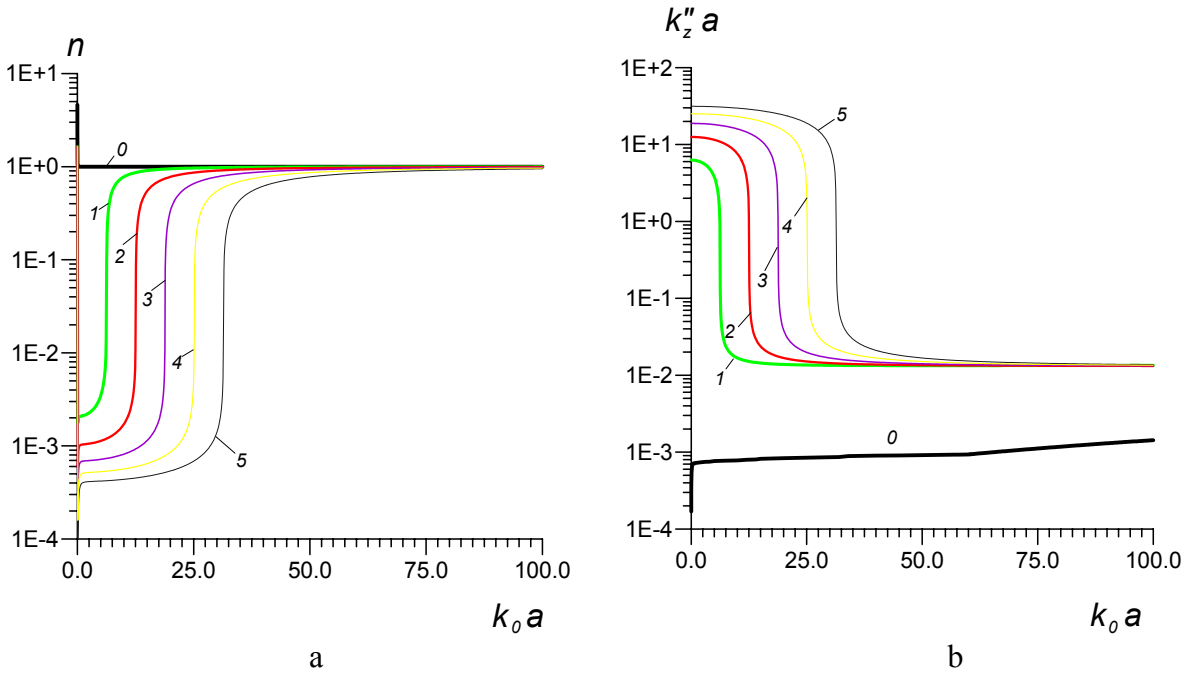


Fig. 5. The deceleration (a) and the normalized loss (b) for even LM_{2m}- modes (the number of curve corresponds to the value m) at small frequencies for copper walls correspond to fig. 4

Conclusions

The nonlinear equations for two-component transverse RDW wavenumber determination have been obtained using dielectric waveguide integral equation and corresponding functionals. The eigenmode dispersion problem is reduced to the solution of two nonlinear equations for two parameters. For this the standard numerical methods may be applied, for example, the iterations procedures or the minimization of discrepancies. Apparently, the last way may be more efficient to determine all dispersion branches, and also for the regions near critical frequencies. The dimension for the quadratic DW with symmetric modes ($\alpha = \beta$) is unit. The derived equations also may be used below the frequency cutoff, and for the lossy dielectric case when the propagation constant γ is complex.

The extension of derived equations for nonuniform in one direction RDW has been performed using the piecewise constant approximation. In this case the problem dimension equals to three, i.e. one must solve three equations with three unknowns.

The field presentation as the sum of functions like (1) with unknown amplitudes and the set of parameters which satisfy (2) may be used in the case of DR with arbitrary boundary. Such functions play a part of metaharmonic functions for field approximation along with cylindrical harmonics [1,17,32].

The on-dimensional case of core coated by finite or infinite multilayered coating has been also investigated numerically using the iteration procedures. The results of complex mode solutions demonstrate some interesting properties at low frequencies.

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THE FORMULATION OF A PROBLEM ABOUT THE POCKELS ELECTROOPTICAL EFFECT IN FORM OF THE MAXWELL EQUATIONS FOR SOME CRYSTALS: LITHIUM NIOBATE, BARIUM TITANATE, STRONTIUM-BARIUM NIOBATE

Yu.A. Zyuryukin, *Member IEEE*, M.V. Pavlova, , *Member IEEE*

Saratov State Technical University, Saratov, Russia
e-mail: phys@sstu.ru

Abstract – The nonconventional problem statement about the electrooptical effect of the Pockels in form of the Maxwell equations for some electrooptical crystals is offered. The generalized expressions, which spot allowances to the components of the impermeability tensor caused by the linear electrooptical effect, are obtained. The expressions for the components of vector of electric intensity of optical plane wave propagating in the given electrooptical crystals, are deduced. The Maxwell scalar equations which have been written down concerning required components of dielectric density and magnetic intensity of optical plane wave, taking into account the Pockels electro-optical effect for crystal of lithium niobate, are gained. The given statement of a problem about the Pockels electro-optical effect in form of the Maxwell equations allows to transfer to the wave equations and to find their solutions for optical waves propagating in presented crystals, taking into account the Pockels electro-optical effect.

1. Introduction

At studying of propagation of electromagnetic waves in anisotropic mediums, including the presence of an external electric field, the method of an ellipsoid of refractive exponents [1] is widely used. The given method allows to determine refractive exponents in the given propagation directions in a crystal, but does not give, in our opinion accurate representation about the transformation of the field structures of electromagnetic waves in the process of their propagation in anisotropic mediums, in particular in electrooptical crystals in the conditions of existence of the Pockels electro-optical effect.

In the present work a problem statement about propagation of optical waves in the some crystals in the presence of the linear electro-optical effect, in form of the Maxwell equations is offered, allowing further to solve a task concerning fields of these waves.

2. The definition of the components of the impermeability tensor of a crystal in the conditions of existence of the Pockels electro-optical effect

The propagation of optical waves in a crystal is identified by the impermeability tensor:

$$\hat{\eta} = \varepsilon_0 \left(\hat{\varepsilon}\right)^{-1}, \tag{1}$$

where $\hat{\varepsilon} = \begin{pmatrix} \varepsilon_{xx}^{o} & 0 & 0 \\ 0 & \varepsilon_{yy}^{o} & 0 \\ 0 & 0 & \varepsilon_{zz}^{o} \end{pmatrix}$ - the permittivity tensor in coordinates system which coincides with main dielectric axes in a crystal (without the Pockels electro-optical effect); ε_0 - a permittivity of vacuum, and we consider, that the crystal is the homogeneous, non-absorptive and magnitno-isotropic medium.

According to a quantum theory of solid bodies, the impermeability tensor depends on allocation of charges in a crystal. Superimposition of exterior electric field $\vec{E}^{\text{ст}}$ will lead to redistribution of latent electricities and to partial deformation of the ionic lattice, what in turn will lead to change of the impermeability tensor:

$$\Delta\eta_{ij} = \eta_{ij}(\vec{E}^{\text{ст}}) - \eta_{ij}(0) = r_{ijk}E_k^{\text{ст}}, \qquad (2)$$

where the summation index k means: $1 = x$, $2 = y$, $3 = z$;

$\hat{\eta}(0) = \hat{\eta}^{\text{o}} = \begin{pmatrix} \eta_{xx}^{\text{o}} & 0 & 0 \\ 0 & \eta_{yy}^{\text{o}} & 0 \\ 0 & 0 & \eta_{zz}^{\text{o}} \end{pmatrix}$ - the impermeability tensor without the Pockels electro-optical effect, and: $\eta_{xx}^{\text{o}} = \dfrac{\varepsilon_0}{\varepsilon_{xx}^{\text{o}}}$, $\eta_{yy}^{\text{o}} = \dfrac{\varepsilon_0}{\varepsilon_{yy}^{\text{o}}}$, $\eta_{zz}^{\text{o}} = \dfrac{\varepsilon_0}{\varepsilon_{zz}^{\text{o}}}$;

$\hat{\eta}(\vec{E}^{\text{ст}}) = \hat{\eta} = \begin{pmatrix} \eta_{xx} & \eta_{xy} & \eta_{xz} \\ \eta_{yx} & \eta_{yy} & \eta_{yz} \\ \eta_{zx} & \eta_{zy} & \eta_{zz} \end{pmatrix}$ - the impermeability tensor in the presence of a stationary exterior electric field $\vec{E}^{\text{ст}}$;

$$\hat{r} = \begin{pmatrix} r_{11,1} & r_{11,2} & r_{11,3} \\ r_{22,1} & r_{22,2} & r_{22,3} \\ r_{33,1} & r_{33,2} & r_{33,3} \\ r_{32,1} & r_{32,2} & r_{32,3} \\ r_{13,1} & r_{13,2} & r_{13,3} \\ r_{12,1} & r_{12,2} & r_{12,3} \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \\ r_{61} & r_{62} & r_{63} \end{pmatrix}$$ - the electro-optical tensor of the third rank.

The electric intensity of optical beam (of electromagnetic plane wave), taking into account the Pockels electro-optical effect, is determined by the vector relation:

$$\vec{E} = \frac{1}{\varepsilon_0}\hat{\eta}\,\vec{D},$$

where $\vec{D}$ - the electric inductance vector, or three scalar expressions which have been written down in the Cartesian coordinates system, at which axes coincide with main dielectric axes in the unperturbed crystal (i.e. without an exterior electric field):

$$\begin{aligned} E_x &= \frac{1}{\varepsilon_0}\left(\eta_{xx}D_x + \eta_{xy}D_y + \eta_{xz}D_z\right), \\ E_y &= \frac{1}{\varepsilon_0}\left(\eta_{yx}D_x + \eta_{yy}D_y + \eta_{yz}D_z\right), \\ E_z &= \frac{1}{\varepsilon_0}\left(\eta_{zx}D_x + \eta_{zy}D_y + \eta_{zz}D_z\right), \end{aligned} \qquad (3)$$

The components of the indignant impermeability tensor in expressions (3) are determined as follows:

$$\begin{aligned} &\eta_{xx} = \eta_{xx}^{\text{o}} + \Delta\eta_{xx}; && \eta_{xy} = \Delta\eta_{xy}; && \eta_{xz} = \Delta\eta_{xz}; \\ &\eta_{yx} = \Delta\eta_{yx}; && \eta_{yy} = \eta_{yy}^{\text{o}} + \Delta\eta_{yy}; && \eta_{yz} = \Delta\eta_{yz}; \\ &\eta_{zx} = \Delta\eta_{zx}; && \eta_{zy} = \Delta\eta_{zy}; && \eta_{zz} = \eta_{zz}^{\text{o}} + \Delta\eta_{zz}. \end{aligned} \qquad (4)$$

The allowances to expressions (4) for a components of the impermeability tensor, caused by electro-optical effect, in a general form are determined on the basis of a relation (2), taking into account a known rule of a relabel of pair coefficients: $(11) \Rightarrow 1$; $(22) \Rightarrow 2$; $(33) \Rightarrow 3$; $(23) = (32) \Rightarrow 4$; $(13) = (31) \Rightarrow 5$; $(12) = (21) \Rightarrow 6$, as follows:

$$\Delta\eta_{11} = r_{11,1}E_x^{\text{ст}} + r_{11,2}E_y^{\text{ст}} + r_{11,3}E_z^{\text{ст}} = r_{11}E_x^{\text{ст}} + r_{12}E_y^{\text{ст}} + r_{13}E_z^{\text{ст}};$$

$$\Delta\eta_{12} = \Delta\eta_{21} = r_{12,1}E_x^{\text{ст}} + r_{12,2}E_y^{\text{ст}} + r_{12,3}E_z^{\text{ст}} = r_{61}E_x^{\text{ст}} + r_{62}E_y^{\text{ст}} + r_{63}E_z^{\text{ст}};$$
$$\Delta\eta_{13} = \Delta\eta_{31} = r_{13,1}E_x^{\text{ст}} + r_{13,2}E_y^{\text{ст}} + r_{13,3}E_z^{\text{ст}} = r_{51}E_x^{\text{ст}} + r_{52}E_y^{\text{ст}} + r_{53}E_z^{\text{ст}};$$
$$\Delta\eta_{22} = r_{22,1}E_x^{\text{ст}} + r_{22,2}E_y^{\text{ст}} + r_{22,3}E_z^{\text{ст}} = r_{21}E_x^{\text{ст}} + r_{22}E_y^{\text{ст}} + r_{23}E_z^{\text{ст}};$$
$$\Delta\eta_{23} = \Delta\eta_{32} = r_{23,1}E_x^{\text{ст}} + r_{23,2}E_y^{\text{ст}} + r_{23,3}E_z^{\text{ст}} = r_{41}E_x^{\text{ст}} + r_{42}E_y^{\text{ст}} + r_{43}E_z^{\text{ст}};$$
$$\Delta\eta_{33} = r_{33,1}E_x^{\text{ст}} + r_{33,2}E_y^{\text{ст}} + r_{33,3}E_z^{\text{ст}} = r_{31}E_x^{\text{ст}} + r_{32}E_y^{\text{ст}} + r_{33}E_z^{\text{ст}}$$

or

$$\begin{aligned}
\Delta\eta_{xx} &= r_{11}E_x^{\text{ст}} + r_{12}E_y^{\text{ст}} + r_{13}E_z^{\text{ст}};\\
\Delta\eta_{xy} = \Delta\eta_{yx} &= r_{61}E_x^{\text{ст}} + r_{62}E_y^{\text{ст}} + r_{63}E_z^{\text{ст}};\\
\Delta\eta_{xz} = \Delta\eta_{zx} &= r_{51}E_x^{\text{ст}} + r_{52}E_y^{\text{ст}} + r_{53}E_z^{\text{ст}};\\
\Delta\eta_{yy} &= r_{21}E_x^{\text{ст}} + r_{22}E_y^{\text{ст}} + r_{23}E_z^{\text{ст}};\\
\Delta\eta_{yz} = \Delta\eta_{zy} &= r_{41}E_x^{\text{ст}} + r_{42}E_y^{\text{ст}} + r_{43}E_z^{\text{ст}};\\
\Delta\eta_{zz} &= r_{31}E_x^{\text{ст}} + r_{32}E_y^{\text{ст}} + r_{33}E_z^{\text{ст}}.
\end{aligned} \tag{5}$$

3. The definition of the components of vector of an electric intensity of optical plane wave, propagating in crystals of lithium niobate, barium titanate, strontium-barium niobate

For the crystal of lithium niobate ($LiNbO_3$), having a symmetry point group 3m, the electro-optical tensor is determined in the form [2]:

$$\hat{r} = \begin{pmatrix} 0 & -r_{22} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ -r_{22} & 0 & 0 \end{pmatrix}, \tag{6}$$

where at optical wave length $\lambda = 0{,}633\,(\text{mcm})$: $r_{13} = 8{,}6\cdot 10^{-12}$ (m/V) , $r_{33} = 30{,}8\cdot 10^{-12}$ (m/V), $r_{22} = 3{,}4\cdot 10^{-12}$ (m/V), $r_{51} = 28\cdot 10^{-12}$ (m/V).

Then allowances to the impermeability tensor, caused by the Pockels electro-optical effect, defined by expressions (5), for crystal $LiNbO_3$ become:

$$\Delta\eta_{xx} = \left(-r_{22}\right)E_y^{\text{ст}} + r_{13}E_z^{\text{ст}}; \qquad \Delta\eta_{xy} = \Delta\eta_{yx} = \left(-r_{22}\right)E_x^{\text{ст}}; \qquad \Delta\eta_{xz} = \Delta\eta_{zx} = r_{51}E_x^{\text{ст}};$$
$$\Delta\eta_{yy} = r_{22}E_y^{\text{ст}} + r_{13}E_z^{\text{ст}}; \qquad \Delta\eta_{yz} = \Delta\eta_{zy} = r_{51}E_y^{\text{ст}}; \ \Delta\eta_{zz} = r_{33}E_z^{\text{ст}}. \tag{7}$$

The components of the vector of electric intensity $\vec{E}$ of optical plane wave, propagating in the given crystal, taking into account the linear electro-optical effect, expressed through the components of the electric inductance vector $\vec{D}$, are defined by substitution of expressions (7) into (4), and then the obtained result - into expressions (3), and become:

$$E_x = \frac{1}{\varepsilon_0}\left(\left\langle \frac{1}{\varepsilon_{xx}^{\text{o}}} + \left(-r_{22}\right)E_y^{\text{ст}} + r_{13}E_z^{\text{ст}} \right\rangle D_x + \left\langle \left(-r_{22}\right)E_x^{\text{ст}} \right\rangle D_y + \left\langle r_{51}E_x^{\text{ст}} \right\rangle D_z \right),$$

$$E_y = \frac{1}{\varepsilon_0}\left(\left\langle \left(-r_{22}\right) E_x^{\text{ст}} \right\rangle D_x + \left\langle \frac{1}{\varepsilon_{yy}^{\text{o}}} + r_{22}E_y^{\text{ст}} + r_{13}E_z^{\text{ст}} \right\rangle D_y + \left\langle r_{51}E_y^{\text{ст}} \right\rangle D_z \right), \tag{8}$$

$$E_z = \frac{1}{\varepsilon_0}\left(\left\langle r_{51}E_x^{\text{ст}} \right\rangle D_x + \left\langle r_{51}E_y^{\text{ст}} \right\rangle D_y + \left\langle \frac{1}{\varepsilon_{zz}^{\text{o}}} + r_{33}E_z^{\text{ст}} \right\rangle D_z \right).$$

For the crystals of barium titanate ($BaTiO_3$) and strontium-barium niobate ($Sr_{0,75}Ba_{0,25}Nb_2O_6$), having a symmetry group *4mm*, the electro-optical tensor is determined in the form:

$$\hat{r} = \begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

where values of electro-optical coefficients for crystal $BaTiO_3$ at optical wave length $\lambda = 0{,}633\,(\text{mcm})$: $r_{13} = 8 \cdot 10^{-12}$ (m/V) , $r_{33} = 28 \cdot 10^{-12}$ (m/V) ; at $\lambda = 0{,}546\,(\text{mcm})$: $r_{51} = 820 \cdot 10^{-12}$ (m/V) . For the crystal $Sr_{0,75}Ba_{0,25}Nb_2O_6$ - at optical wave length $\lambda = 0{,}633\,(\text{mcm}) : r_{13} = 67 \cdot 10^{-12}$ (m/V) , $r_{33} = 1640 \cdot 10^{-12}$ (m/V) , $r_{51} = 42 \cdot 10^{-12}$ (m/V) .

In this case, by analogy, expressions (5) for allowances to the impermeability tensor will become:

$$\begin{gathered} \Delta\eta_{xx} = r_{13}E_z^{\text{ст}}; \qquad \Delta\eta_{xy} = \Delta\eta_{yx} = 0; \qquad \Delta\eta_{xz} = \Delta\eta_{zx} = r_{51}E_x^{\text{ст}}; \\ \Delta\eta_{yy} = r_{13}E_z^{\text{ст}}; \qquad \Delta\eta_{yz} = \Delta\eta_{zy} = r_{51}E_y^{\text{ст}}; \qquad \Delta\eta_{zz} = r_{33}E_z^{\text{ст}}, \end{gathered} \tag{10}$$

and expressions (3) for the crystals $BaTiO_3$ or $Sr_{0,75}Ba_{0,25}Nb_2O_6$, in turn, we will write down as follows:

$$\begin{gathered} E_x = \frac{1}{\varepsilon_0}\left(\left\langle \frac{1}{\varepsilon_{xx}^{\text{o}}} + r_{13}E_z^{\text{ст}} \right\rangle D_x + \left\langle r_{51}E_x^{\text{ст}} \right\rangle D_z \right), \\ E_y = \frac{1}{\varepsilon_0}\left(\left\langle \frac{1}{\varepsilon_{yy}^{\text{o}}} + r_{13}E_z^{\text{ст}} \right\rangle D_y + \left\langle r_{51}E_y^{\text{ст}} \right\rangle D_z \right), \\ E_z = \frac{1}{\varepsilon_0}\left(\left\langle r_{51}E_x^{\text{ст}} \right\rangle D_x + \left\langle r_{51}E_y^{\text{ст}} \right\rangle D_y + \left\langle \frac{1}{\varepsilon_{zz}^{\text{o}}} + r_{33}E_z^{\text{ст}} \right\rangle D_z \right). \end{gathered} \tag{11}$$

4. The Maxwell equations for optical plane waves propagating in the crystals, taking into account the Pockels electro-optical effect

Now we will be turned to system of the homogeneous Maxwell equations, which completely describe an electromagnetic field of optical wave propagating in a crystal. Two first Maxwell equations:

$$\text{rot}\,\vec{H} = \frac{\partial \vec{D}}{\partial t}, \tag{12}$$

$$\operatorname{rot}\vec{E} = -\mu\mu_0 \frac{\partial \vec{H}}{\partial t}, \tag{13}$$

we will write down in the scalar form in the Cartesian coordinates system. Then the first Maxwell equation (12) becomes:

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\partial D_x}{\partial t}, \quad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{\partial D_y}{\partial t}, \quad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{\partial D_z}{\partial t}, \tag{14}$$

and accordingly the second equation (13) becomes:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu\mu_0 \frac{\partial H_x}{\partial t}, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu\mu_0 \frac{\partial H_y}{\partial t}, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu\mu_0 \frac{\partial H_z}{\partial t}, \tag{15}$$

where $\vec{H}$ - magnetic intensity of optical plane wave, propagating in a crystal, μ - magnetic permeability of medium, μ_0 - permeability of vacuum.

Obtained above expression (8) for the components of an electric intensity of the wave propagating in crystal $LiNbO_3$, taking into account the Pockels electro-optical effect, we will substitute in the equations (15). In this case the equations (15) become:

$$\begin{aligned} &\left\langle r_{51}E_x^{\text{ст}}\right\rangle \frac{\partial D_x}{\partial y} + \left\langle r_{51}E_y^{\text{ст}}\right\rangle \frac{\partial D_y}{\partial y} + \left\langle \frac{1}{\varepsilon_{zz}^{\text{o}}} + r_{33}E_z^{\text{ст}}\right\rangle \frac{\partial D_z}{\partial y} - \left\langle \left(-r_{22}\right)E_x^{\text{ст}}\right\rangle \frac{\partial D_x}{\partial z} - \\ &- \left\langle \eta_{yy}^{\text{o}} + r_{22}E_y^{\text{ст}} + r_{13}E_z^{\text{ст}}\right\rangle \frac{\partial D_y}{\partial z} - \left\langle r_{51}E_y^{\text{ст}}\right\rangle \frac{\partial D_z}{\partial z} = -\mu\mu_0\varepsilon_0 \frac{\partial H_x}{\partial t}; \end{aligned} \tag{16}$$

$$\begin{aligned} &\left\langle \eta_{xx}^{\text{o}} + \left(-r_{22}\right)E_y^{\text{ст}} + r_{13}E_z^{\text{ст}}\right\rangle \frac{\partial D_x}{\partial z} + \left\langle \left(-r_{22}\right)E_x^{\text{ст}}\right\rangle \frac{\partial D_y}{\partial z} + \left\langle r_{51}E_x^{\text{ст}}\right\rangle \frac{\partial D_z}{\partial z} - \\ &- \left\langle r_{51}E_x^{\text{ст}}\right\rangle \frac{\partial D_x}{\partial x} - \left\langle r_{51}E_y^{\text{ст}}\right\rangle \frac{\partial D_y}{\partial x} - \left\langle \frac{1}{\varepsilon_{zz}^{\text{o}}} + r_{33}E_z^{\text{ст}}\right\rangle \frac{\partial D_z}{\partial x} = -\mu\mu_0\varepsilon_0 \frac{\partial H_y}{\partial t}; \end{aligned} \tag{17}$$

$$\begin{aligned} &\left\langle \left(-r_{22}\right)E_x^{\text{ст}}\right\rangle \frac{\partial D_x}{\partial x} + \left\langle \eta_{yy}^{\text{o}} + r_{22}E_y^{\text{ст}} + r_{13}E_z^{\text{ст}}\right\rangle \frac{\partial D_y}{\partial x} + \left\langle r_{51}E_y^{\text{ст}}\right\rangle \frac{\partial D_z}{\partial x} - \\ &- \left\langle \eta_{xx}^{\text{o}} + \left(-r_{22}\right)E_y^{\text{ст}} + r_{13}E_z^{\text{ст}}\right\rangle \frac{\partial D_x}{\partial y} - \left\langle \left(-r_{22}\right)E_x^{\text{ст}}\right\rangle \frac{\partial D_y}{\partial y} - \left\langle r_{51}E_x^{\text{ст}}\right\rangle \frac{\partial D_z}{\partial y} = -\mu\mu_0\varepsilon_0 \frac{\partial H_z}{\partial t}. \end{aligned} \tag{18}$$

5. Conclusions

Thus, have gained six scalar Maxwell equations ((14) and (16) - (18)), which are written down concerning required the components of the electric inductance D_x, D_y, D_z and magnetic intensity H_x, H_y, H_z of the optical plane wave, taking into account the Pockels electro-optical effect, for lithium niobate. For other crystals ($BaTiO_3$, $Sr_{0,75}Ba_{0,25}Nb_2O_6$), which considered here, the similar equations are obtained by analogy by substitution of corresponding expressions - (11) - for the components of electric intensity, into the second Maxwell equation (15) in the coordinate form.

Then at the given statement of a problem about the Pockels electro-optical effect in form of the Maxwell equations, it is possible to transfer to the wave equations, having excluded H_x, H_y, H_z (having substituted (16) - (18) into (14)) and to find their solutions, for example, for the optical plane waves, propagating in considered crystals, taking into

account the Pockels electro-optical effect. Moreover, using a method of transformation of coordinates, it is possible to gain the expressions determining phase velocities of the optical waves propagating in electro-optical crystals in any direction.

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APPROXIMATE ANALYTICAL MODELS FOR CALCULATION OF CUTOFF WAVELENGTHS OF SOME HIGH ORDER MODES OF COMPLEX CROSS SECTION SHAPED WAVEGUIDES

V.V. Komarov, *Member IEEE*, A.G. Savina, I.I. Novruzov

Saratov State Technical University, Russia
e-mail: vkom@mail.saratov.ru

Abstract – Cutoff wavelengths of some high-order modes propagating in complex cross section shaped waveguides are simulated using approximate analytical models taken from the literature. Obtained results are compared with numerical data.

1. Introduction

Complex cross section shaped waveguides (CCSW) find application in modern microwave engineering as basic units of directional couplers, transitions, ferrite circulators, polarizers and etc. Most of such waveguides belong to one of two groups: so-called waveguides with capacitance gap (WCG) and waveguides without metal septa. Rectangular T-septum waveguide (RTSW) and horseshoe shaped waveguide (HSW) are the examples of WCG (fig.1,a,b). Crossed rectangular waveguide (CRW) can be included in the second group (fig.1,c).

Different numerical approaches: partial domains method (PDM), method of integral equations, finite-difference method and finite element method (FEM) are used for computation of CCSW. Along with numerical techniques several analytical approximations sometimes are applied for the same purpose. Homogeneous and inhomogeneous CCSW can be simulated using transverse resonance method, perturbation method, equivalent networks method (ENM).

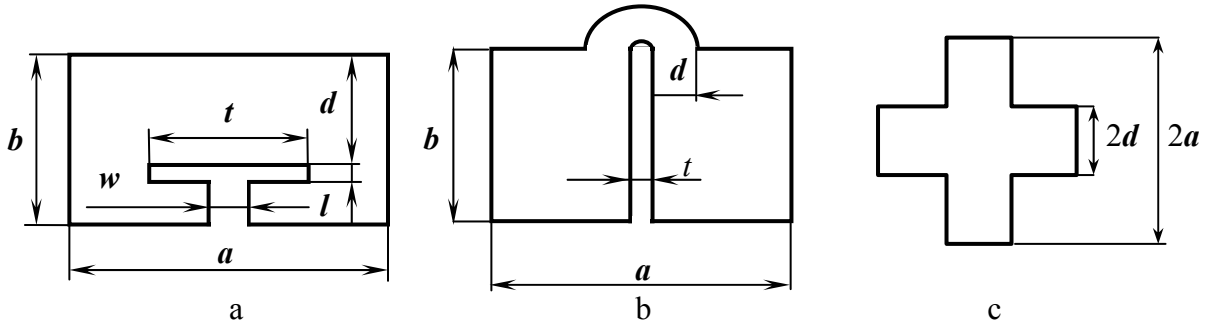

Fig. 1. Rectangular T-septum waveguide (a); horseshoe shaped waveguide (b) and crossed waveguide (c)

2. Computational models for waveguides with capacitance gap

It is well known that the dominant quasi-H_{10} mode of any WCG can be analyzed by means of ENM [1]. One of the restrictions of given method is impossibility of modeling high-order modes. We propose Oliner model, which is utilized presently for simulation of high-order modes in microsrtip lines [2] for calculation of the first high-order mode in RTSW:

$$\lambda_{C2} = 2t_0\sqrt{\varepsilon'}\ ; \quad 2 \le \varepsilon' \le 8\ , \tag{1}$$

$$\lambda_{C2} = 2t_0\sqrt{\varepsilon^*}\ ; \quad 9 \le \varepsilon' \le 81\ , \tag{2}$$

$$t_0 = t + \frac{l}{\pi}\left(\ln\left(\frac{2d}{l}\right) + 1\right)\ , \tag{3}$$

$$\varepsilon^* = 0.5(\varepsilon' + 1) + \frac{0.5(\varepsilon' - 1)}{\sqrt{1 + \frac{10d}{l}}}\ , \tag{4}$$

where ε′ is the dielectric permittivity of the capacitance gap filling; *t,d,l* are the sizes of RTSW (fig.1,a).

The first high-order mode in HSW can be approximately determined with the help of the expression for H_{11}-mode in coaxial line [3]:

$$\lambda_{C2} = 2\pi r_m\left[1 + 0.16666\left(\frac{u_m}{2r_m}\right)^2 - 0.058333\left(\frac{u_m}{2r_m}\right)^4\right], \tag{5}$$

where $r_m = r_к + R_к$ и $u_m = 2(R_к - r_к)$; here $R_к$ и $r_к$ are the radiuses of the internal and external conductors of coaxial line. Sizes of coaxial line and HSW (fig.1,b):

$$2R_к = t + 2d;\ \ 2r_к = t;\ r_m = t + d;\ u_m = 2d\ . \tag{6}$$

When capacitance gap of HSW is filled with dielectric the cutoff wavelength of the first high-order mode HE_2 is defined as:

$$\lambda_{C2} = k_\lambda \pi\sqrt{\varepsilon}(t + d)\left[1 + 0.16666\left(\frac{d}{t+d}\right)^2 - 0.058333\left(\frac{d}{t+d}\right)^4\right]. \tag{7}$$

Results of analytical and numerical modeling of the first high order mode in RTSW and HSW with dielectric in capacitance gap are represented in fig.2 and 3 respectively. Empirical coefficient $k_\lambda = 0.85$ was used in expression (7) for calculations of HSW. Computational error for RTSW with sizes $0.3 \le t/a \le 0.9$ does not exceed $\Delta \le 4.5\%$ but when $0.1 \le t/a \le 0.3$ it rises up to $\Delta\% = 5 \div 12$. Approximate analytical model for HSW shows $\Delta = 6\%$ when $t/a < 0.25$ and $\Delta = 8\ \%$ when $0.25 \le t/a \le 0.4$ relatively FEM. Oliner model was compared with the Ritz-Galerkin method employed in [4] for simulation of inhomogeneous RTSW.

3. Computational model for the lowest E-mode in crossed waveguide

Wave numbers of the lowest E-mode in CCSW of the second group can be obtained with the help of approximate model, which takes into account some geometrical parameters of a waveguide cross section [5]:

$$\frac{2\pi}{\lambda_C} = \nu\left[\frac{\pi L}{2qS^2}\right]^{0.25}, \qquad (8)$$

where ν = 2.4049 is the first root of Bessel function; L is the waveguide perimeter; S is the waveguide cross section square; q is the radius of inscribed circle.

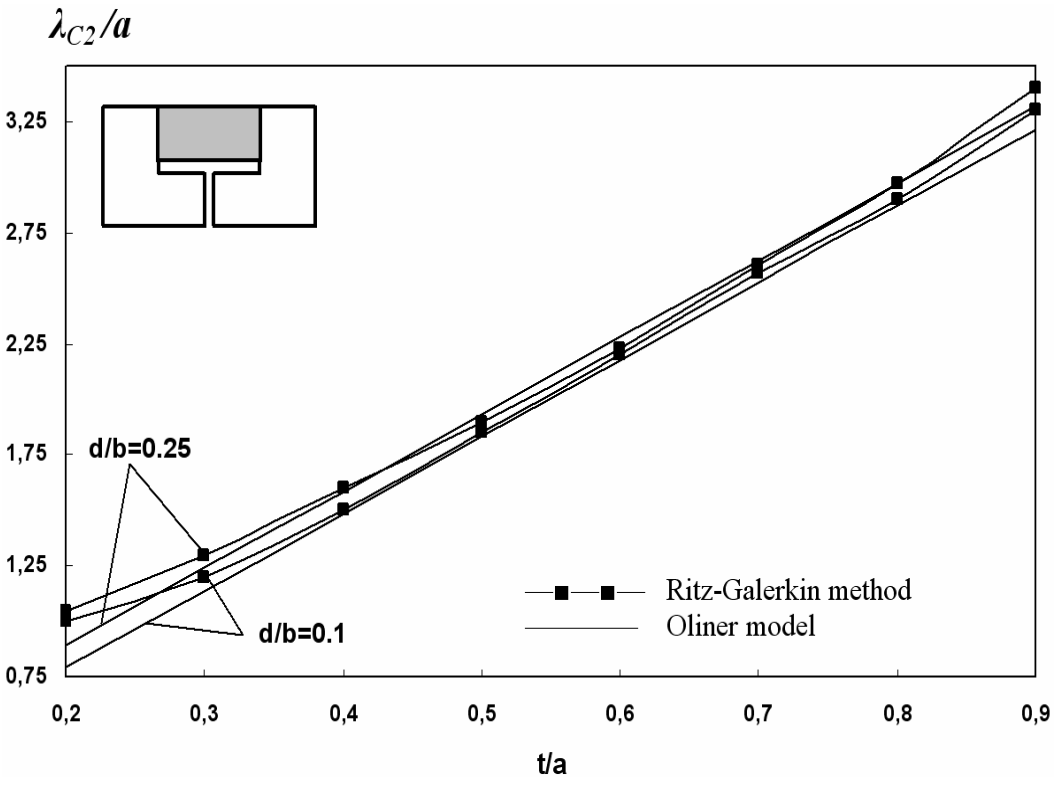

Fig. 2. Cutoff wavelength of the HE_2 mode in RTSW partially loaded with dielectric: *b/a*= 0.45*; l/b = 0.05;* ε′ = 3

Expression (8) has been tested in [5] on an examples of elliptical, sector, triangle and some other waveguides. We employed this model for calculation of cutoff wavelengths of E_{11} mode propagating in symmetrical CRW with $L = 8a$; $S = 8ad - 4d^2$. Comparison with theoretical data obtained by means of PDM [6] and FEM [7] shows a good agreement between all three approaches (fig.4).

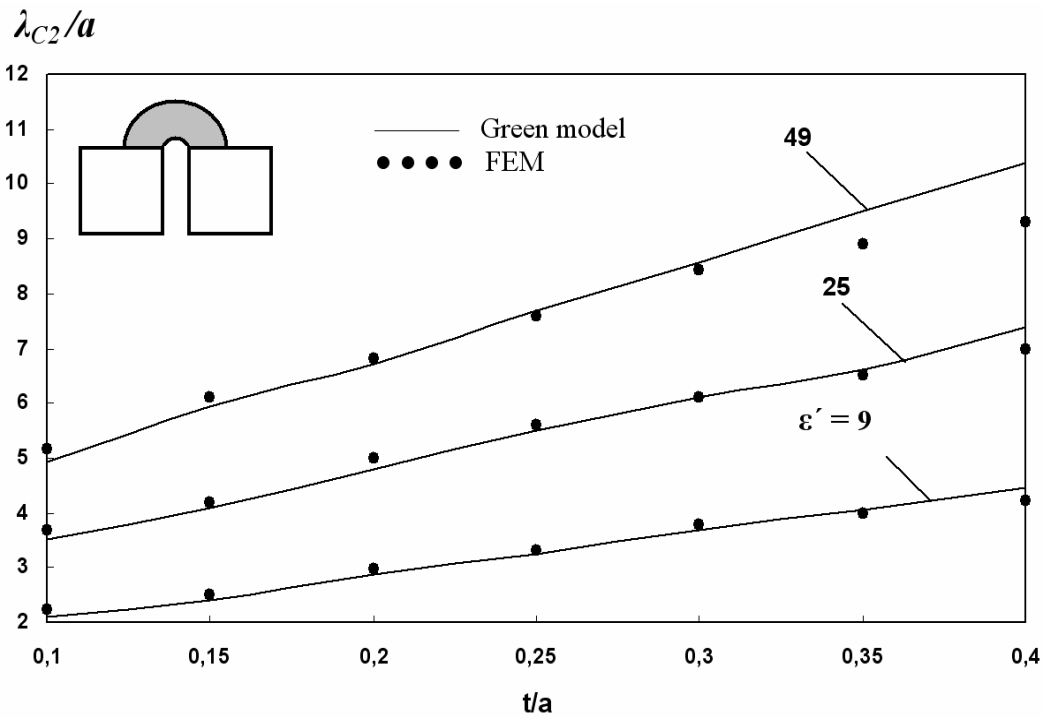

Fig. 3. Cutoff wavelength of the HE_2 mode in HSW partially loaded with dielectric: b/a= 0.5; d/a = 0.15

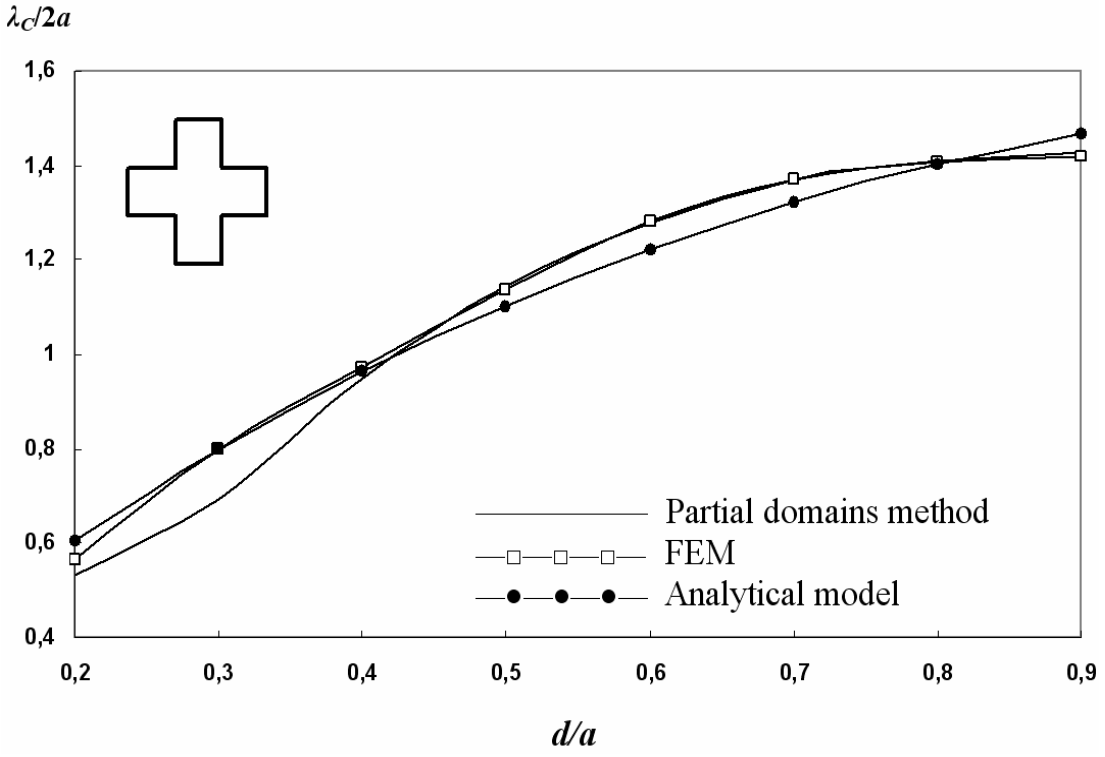

Fig. 4. Cutoff wavelengths of E_{11} mode in CRW

4. Conclusion

So described analytical models can be successfully adapted for approximate calculation of some high-order H-, E- and HE-modes in different type CCSW.

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VECTOR ELECTRIC AND MAGNETIC POTENTIALS IN ELECTRODYNAMICS OF CONTINUOUS MEDIA

M.V. Davidovich, *Senior member IEEE*

Saratov State University, 410012 Saratov, Russia
E-mail: DavidovichMV@info.sgu.ru

Abstract – The possible approaches for introduction of vector-potentials in inhomogeneous anisotropic and bianisotropic media have been considered, and the new differential and integrodifferential equations for these potentials have been formulated. It has been shown that the potential boundary problem formulations have the advantages instead of field formulations. The iteration algorithms and finite elements have been proposed for solutions of potential boundary problems. The numerical simulations have been performed for investigation of $H_{01\delta}$ oscillations of homogeneous and inhomogeneous dielectric resonators.

Introduction

In classical electrodynamics the final relations and equations, as a rule, are set in form of fields, and the potentials are usually introduced for simplification of problem solutions and for more simple expressions [1-25]. And at that one may do not use any potentials. In classical microscopic electromagnetics (electrodynamics of vacuum) the electrical vector-potential $\vec{A}$ and the scalar one ϕ are connected with the fields by the relations [1]

$$\vec{E} = -\partial\vec{A}/\partial t - grad\phi\,, \quad \vec{B} = \mu_0\vec{H} = rot\vec{A}\,, \tag{1}$$

at that we consider the movement of point (elementary) charges or charged particles under the influence of field and also its excitation by accelerated moving charges. It is well known that the fields completely do not describe the quantum system, and the introduction of potentials is the necessity. In quantum electrodynamics they form the 4-potential and have the clear physical meaning. In macroscopic electromagnetics the vector and scalar potentials also are the measurable values [10,22,26,27], i.e. present the physical reality. The introduction only the electrical (one vector and one scalar) potentials is the consequence of the fact that predicted by Dirac elementary particles with magnetic charges (magnetic monopoles) have not discovered yet.

The classical electrodynamics of continuous media ever operates with the material parameters introducing by averaging over the physical infinitesimal volume (i.e. by the homogenization) of electrical and magnetic polarization inputs under the movement of great number of matter particles in the field. The motion equations are replaced by equivalent in a certain sense material relations, and the field presentation by (1) is not quite convenient, as there are the electrical and magnetic polarization currents of matter. Therefore, in contrast to (1), it is convenient to use two symmetrical defined vector-potentials: the electric one and the magnetic one [28], and also it is useful to introduce two densities of electrical $\vec{j}_{in}^{\,e}$ and magnetic $\vec{j}_{in}^{\,m}$ incident currents, which create the field. The last one is the auxiliary and having the property $div\vec{j}_{in}^{\,m} = 0$ i.e. may be consider (accurate within a factor) as the rotor of some real electric current density. We will consider several approaches to introduction of vector-potentials which describe stationary (harmonic in time) fields with the time dependence $\exp(j\omega t)$. All material parameters and field vectors

may depend on frequency (the time dispersion). We will include the conductivity currents into media permittivity and permeability [28]:

$$\widetilde{\varepsilon} = \hat{\varepsilon} - j\sigma^e/(\omega\varepsilon_0) \quad , \quad \widetilde{\mu} = \hat{\mu} - j\sigma^m/(\omega\mu_0) \quad . \tag{2}$$

The tensors $\hat{\varepsilon}$ and $\hat{\mu}$ will be complex for dispersive medium if even the free electric charges are absent [29]. The complex and frequency-dependent will be also (owing to Drude's formula) the electrical conductivity $\sigma^e = \omega_{ep}^2\varepsilon_0/(j\omega + \omega_{ec})$ caused by such charges (here ω_{ep} is plasma frequency, and ω_{ec} is the free charges collision frequency). The parameters in the relations (2) will be in general complex and inhomogeneous. The relation $div(\widetilde{\mu}\vec{H}) = 0$ is also fulfilled und the condition $div\vec{j}_{in}^{\,m} = 0$. It means the physical absence of volumetric magnetic charges. We will introduce below electric and magnetic polarization current densities. The availability of inhomogeneous magnetic medium leads to appearance of volumetric magnetic charges, as $div((\widetilde{\mu} - \hat{I})\vec{H}) = -div\vec{H} \neq 0$ (here $\hat{I}$ is the unit tensor). The fictitious surface magnetic charges may appear and on the media bedding interfaces [28]. In case of open magnetodielectric resonator it gives the input to magnetic-dipole radiation. It is easy to see that the full magnetic charge of any magnetodielectric body at that is zero. We will further also introduce the general material relations for bianisotropic media (metamaterials), which are characterized by two additional cross-polarization tensors кросс-поляризаций. The material parameters for such artificial media are derived using the homogenization procedure [30,31] which is polysemantic in general case. In such media the material parameter tensors do not commute in general, that may fulfill in magnetic semiconductors in magnetic field. In particular, one can consider the metamaterials with magnetic and dielectric inclusions (including semiconductive ones) into background (matrix), which are placed in constant magnetic field and having two turned on arbitrary angles lattice and even having the different periods. The tensors are non-hermitian due to loss. Below it has been only proposed that they are non-singular.

1. Potentials in inhomogeneous and anisotropic media

By virtue of Helmholtz circulation theorem [32,33] the vector fields are the sum of their potential and solenoidal (vortical) parts and may be presented by several forms, for example, as

$$\vec{E} = -j\omega\mu_0\widetilde{\mu}\vec{A}^e + (j\omega\varepsilon_0)^{-1} grad \bullet div(\widetilde{\varepsilon}^{-1}\vec{A}^e) - rot\vec{A}^m \quad , \tag{3}$$

$$\vec{H} = -j\omega\varepsilon_0\widetilde{\varepsilon}\vec{A}^m + (j\omega\mu_0)^{-1} grad \bullet div(\widetilde{\mu}^{-1}\vec{A}^m) + rot\vec{A}^e \quad . \tag{4}$$

Here the vector-potentials satisfy second order equations which are getting after the relations (3) and (4) substitutions into the Maxwell equations:

$$\left[\widetilde{\varepsilon} grad \bullet div(\widetilde{\varepsilon}^{-1}\vec{A}^e) - rot \bullet rot\vec{A}^e + k_0^2\widetilde{\varepsilon}\widetilde{\mu}\vec{A}^e\right] = -\vec{j}_{in}^{\,e} + j\omega\varepsilon_0(\widetilde{\varepsilon} rot\vec{A}^m - rot\widetilde{\varepsilon}\vec{A}^m) \, , \tag{5}$$

$$\left[\widetilde{\mu} grad \bullet div(\widetilde{\mu}^{-1}\vec{A}^m) - rot \bullet rot\vec{A}^m + k_0^2\widetilde{\mu}\widetilde{\varepsilon}\vec{A}^m\right] = -\vec{j}_{in}^{\,m} + j\omega\mu_0(\widetilde{\mu} rot\vec{A}^e - rot\widetilde{\mu}\vec{A}^e) . \tag{6}$$

The symbol $\bullet$ here denotes the product of operators and it also be omitted, $k_0^2 = \omega^2\varepsilon_0\mu_0$ is the wavenumber. Correspondingly in (5) and (6) the operators may be written in the forms: $rot \bullet rot = \nabla\times\nabla\times$ и $grad \bullet div = \nabla\otimes\nabla = \nabla\nabla\cdot$ Simple point means here the scalar product of two vectors, the symbol $\times$ denotes the vector product, and the symbol $\otimes$ designates their tensor (dyadic) product. Accordingly, for any vector $\vec{a}$ one has $(grad \bullet div)\vec{a} = (\nabla\otimes\nabla)\vec{a} = \nabla(\nabla\cdot\vec{a})$. Further we will use the different forms of operator notation. The operators in the square brackets in (5) and (6), generally speaking, are different. They will be identical for homogeneous anisotropic media if the matrixes $\widetilde{\varepsilon}$ and $\widetilde{\mu}$ commute. Then we get the inhomogeneous general Helmholtz equation

$$\left[\nabla^2 + k_0^2 \tilde{\varepsilon}\tilde{\mu}\right]\vec{A}^{(e,m)} = -\vec{j}_{in}^{(e,m)} \quad . \tag{7}$$

The equations (5) and (6) in general case have not separated. It is necessary for such separation the conversion to zero of second terms in its right parts. One may set up a correspondence for operator *rot* the following matrix or operator

$$\overline{\overline{rot}} = \begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & -\partial/\partial x \\ -\partial/\partial y & \partial/\partial x & 0 \end{bmatrix}.$$

Under the commutation of operator $\overline{\overline{rot}}$ with the matrix $\tilde{\varepsilon}$ and $\tilde{\mu}$ on the vector-potential functions equations are separable. Particularly, for the scalar permittivity and permeability the expressions in the parenthesis in (5), (6) have correspondingly the shape $-\nabla\tilde{\varepsilon}\times\vec{A}^m$ и $\nabla\tilde{\mu}\times\vec{A}^e$, therefore the separation takes place if the vectors $\nabla\tilde{\mu}$, $\vec{A}^e$ and $\nabla\tilde{\varepsilon}$, $\vec{A}^m$ are mutually collinear. For isotropic media (described by scalar parameters) it is possible also the representation in form

$$\vec{E} = -j\omega\mu_0\tilde{\mu}\vec{\vec{A}}^e + \left(j\omega\varepsilon_0\right)^{-1} grad\tilde{\varepsilon}^{-1} div\vec{\vec{A}}^e - rot\vec{\vec{A}}^m \quad , \tag{8}$$

$$\vec{H} = -j\omega\varepsilon_0\tilde{\varepsilon}\vec{\vec{A}}^m + \left(j\omega\mu_0\right)^{-1} grad\tilde{\mu}^{-1} div\vec{\vec{A}}^m + rot\vec{\vec{A}}^e \quad , \tag{9}$$

where the new vector-potentials satisfy the equations

$$\left[\tilde{\varepsilon} grad\tilde{\varepsilon}^{-1} div - rot \bullet rot + k_0^2\tilde{\varepsilon}\tilde{\mu}\right]\vec{\vec{A}}^e = -\vec{j}_{in}^e - j\omega\varepsilon_0\nabla\tilde{\varepsilon}\times\vec{\vec{A}}^m \quad , \tag{10}$$

$$\left[\tilde{\mu} grad\tilde{\mu}^{-1} div - rot \bullet rot + k_0^2\tilde{\mu}\tilde{\varepsilon}\right]\vec{\vec{A}}^m = -\vec{j}_{in}^m + j\omega\mu_0\nabla\tilde{\mu}\times\vec{\vec{A}}^e . \tag{11}$$

The separation here takes place at $\nabla\tilde{\varepsilon}\times\vec{\vec{A}}^m = 0$, $\nabla\tilde{\mu}\times\vec{\vec{A}}^e = 0$, and the equations get the sufficiently simple forms, for example $\left[\nabla^2 + k_0^2\tilde{\varepsilon}\tilde{\mu}\right]\vec{\vec{A}}^e + \left(\tilde{\varepsilon}\nabla\tilde{\varepsilon}^{-1}\right)\nabla\cdot\vec{\vec{A}}^e = -\vec{j}_{in}^e$. At last, the fields may be presented in the following way:

$$\vec{E} = -j\omega\mu_0\vec{\tilde{A}}^e + \left(j\omega\varepsilon_0\right)^{-1} grad \bullet div\vec{\tilde{A}}^e - rot\vec{\tilde{A}}^m \quad , \tag{12}$$

$$\vec{H} = -j\omega\varepsilon_0\vec{\tilde{A}}^m + \left(j\omega\mu_0\right)^{-1} grad \bullet div\vec{\tilde{A}}^m + rot\vec{\tilde{A}}^e \quad . \tag{13}$$

And in this case the equations for introduces in (12), (13) potentials become so:

$$\left(k_0^2 + \nabla^2\right)\vec{\tilde{A}}^e = -\tilde{\varepsilon}^{-1}\vec{j}_{in}^e + \left(\tilde{\varepsilon}^{-1} - \hat{I}\right)\left[\nabla\times\nabla\times\vec{\tilde{A}}^e - j\omega\varepsilon_0\nabla\times\vec{\tilde{A}}^m\right] , \tag{14}$$

$$\left(k_0^2 + \nabla^2\right)\vec{\tilde{A}}^m = -\tilde{\mu}^{-1}\vec{j}_{in}^m + \left(\tilde{\mu}^{-1} - \hat{I}\right)\left[\nabla\times\nabla\times\vec{\tilde{A}}^m + j\omega\mu_0\nabla\times\vec{\tilde{A}}^e\right] . \tag{15}$$

They may be written in this way:

$$\left(k_0^2 + \nabla^2\right)\vec{\tilde{A}}^e = -\vec{j}_{in}^e - \left(\tilde{\varepsilon} - \hat{I}\right)\left[k_0^2\vec{\tilde{A}}^e + \nabla\nabla\cdot\vec{\tilde{A}}^e - j\omega\varepsilon_0\nabla\times\vec{\tilde{A}}^m\right],$$

$$\left(k_0^2 + \nabla^2\right)\vec{\tilde{A}}^m = -\vec{j}_{in}^m - \left(\tilde{\mu} - \hat{I}\right)\left[k_0^2\vec{\tilde{A}}^m + \nabla\nabla\cdot\vec{\tilde{A}}^m + j\omega\mu_0\nabla\times\vec{\tilde{A}}^e\right].$$

These equations also are non-separable разделяются. The partial separation takes place either for nonmagnetic ($\tilde{\mu} = \hat{I}$), or for nonelectric ($\tilde{\varepsilon} = \hat{I}$) media. Inn the first case the magnetic vector-potential satisfy Helmholtz equation $\left(\nabla^2 + k_0^2\right)\vec{\tilde{A}}^m = -\vec{j}_{in}^m$, which has the standard solution. Using this solution in the right part of (14), one may get the solution for $\vec{\tilde{A}}^e$. If one considers the potential vectors $\vec{\tilde{A}}^{(e,m)}$, that this case leads to potential fields, that is the trivial, and the case $\tilde{\varepsilon} = \tilde{\mu} = \hat{I}$ is also trivial. besides the mentioned vector-potentials one may use some others vector-potential functions, for example, $\vec{A}'^e = \tilde{\mu}^{-1}\vec{\tilde{A}}^e$,

$\vec{A}'^m = \tilde{\varepsilon}^{-1}\vec{\tilde{A}}^m$. This becomes clear by ambiguity of potentials in the electrodynamics of continuum [10].

In the simplest case coordinate-independent scalar material parameters the equations (3) and (4) are simplified [28]:

$$\vec{E} = \frac{\nabla\otimes\nabla + k_0{}^2\tilde{\varepsilon}\tilde{\mu}}{j\omega\varepsilon_0\tilde{\varepsilon}}\vec{A}^e - rot\vec{A}^m \quad , \tag{16}$$

$$\vec{H} = \frac{\nabla\otimes\nabla + k_0{}^2\tilde{\varepsilon}\tilde{\mu}}{j\omega\mu_0\tilde{\mu}}\vec{A}^m + rot\vec{A}^e \quad , \tag{17}$$

in which connection the vector-potentials satisfy the equations (7). To get their solution one can with the help of scalar Green's function (GF) [28] $G(k,\vec{r}) = (4\pi|\vec{r}|)^{-1}\exp(-jk|\vec{r}|)$, where $k = k_0\sqrt{\tilde{\varepsilon}\tilde{\mu}}$. The explotable here electric vector-potential differs from respective potential in the equation (1) by factor $\mu_0\tilde{\mu}$. This is connected with the tradition to use the magnetic filed but not the induction in applied electromagnetic, and also with the conveniences (though the power magnetic vector is the vector $\vec{B}$ [1]). Mark that the fields satisfy the same wave equations but with others more complicated right parts even in for homogeneous and isotropic media [28]:

$$(\nabla^2 + k_0^2)\vec{E} = -j\omega\mu_0\tilde{\mu}\vec{j}^e_{in} + \frac{\nabla\nabla\cdot\vec{j}^e_{in}}{j\omega\varepsilon_0\tilde{\varepsilon}} - \nabla\times\vec{j}^m_{in} \quad , \quad (\nabla^2 + k_0^2)\vec{H} = -j\omega\varepsilon_0\tilde{\varepsilon}\vec{j}^m_{in} + \frac{\nabla\nabla\cdot\vec{j}^m_{in}}{j\omega\mu_0\tilde{\mu}} + \nabla\times\vec{j}^e_{in} \; . \tag{18}$$

This just explains the advantage of usage the vector-potentials. But durante absentia of any separation this advantage is insubstantial. Note also that here owing to Loorentz calibration there is the bond of vector and scalar potentials

$$div\vec{A}^e + j\omega\varepsilon_0\tilde{\varepsilon}\phi^e = 0 \quad , \; div\vec{A}^m + j\omega\mu_0\tilde{\mu}\phi^m = 0 \, ,$$

and the last ones are also satisfy the wave equations

$$(\nabla^2 + k_0^2\tilde{\varepsilon}\tilde{\mu})\phi^e = -(\varepsilon_0\tilde{\varepsilon})^{-1}\rho^e_{in} \quad , \; (\nabla^2 + k_0^2\tilde{\varepsilon}\tilde{\mu})\phi^m = -(\mu_0\tilde{\mu})^{-1}\rho^m_{in} \quad . \tag{19}$$

In general case the fields satisfy following inhomogeneous equations:

$$[-rot\tilde{\mu}^{-1}rot + k_0^2\tilde{\varepsilon}]\vec{E} = -j\omega\mu_0\vec{j}^e_{in} - rot\tilde{\mu}^{-1}\vec{j}^m_{in} \, ,$$

$$[-rot\tilde{\varepsilon}^{-1}rot + k_0^2\tilde{\mu}]\vec{H} = -j\omega\varepsilon_0\vec{j}^m_{in} + rot\tilde{\varepsilon}^{-1}\vec{j}^e_{in} \, .$$

For solving of different problems it is convenient to use also others potentials. These are the Hertz vectors, Borgnis functions, Debye potentials [2]. At that time often the direction of mensionad vector-potentials is fixed (for example, along some coordinate axis which does not change it), or they are directed along ort-vector which changes its direction (for the Debye potentials). Let consider the different Hertz vectors modifications. They are defined standardly so: $\vec{A}^e = j\omega\varepsilon_0\tilde{\varepsilon}\vec{\Pi}^e$, $\vec{A}^m = j\omega\mu_0\tilde{\mu}\vec{\Pi}^m$, and the fields are expressed through it as

$$\vec{E} = (\nabla\otimes\nabla + k_0^2\tilde{\mu}\tilde{\varepsilon})\vec{\Pi}^e - j\omega\mu_0\nabla\times(\tilde{\mu}\vec{\Pi}^m) \quad , \tag{20}$$

$$\vec{H} = (\nabla\otimes\nabla + k_0^2\tilde{\varepsilon}\tilde{\mu})\vec{\Pi}^m + j\omega\varepsilon_0\nabla\times(\tilde{\varepsilon}\vec{\Pi}^e) \quad . \tag{21}$$

In the case of nonstationary fields the multiplication on $(j\omega)^{-1}$ corresponds to integration over the time, and the bond of fields with the potentials becomes integral, and so the Hertz vectors is more preferable at that time. The hertz vectors satisfy the following inhomogeneous wave equations:

$$[k_0^2\tilde{\mu} + \nabla\otimes\nabla\tilde{\varepsilon}^{-1} - \tilde{\varepsilon}^{-1}\nabla\times\nabla\times]\tilde{\varepsilon}\vec{\Pi}^e = -\frac{\tilde{\varepsilon}^{-1}\vec{j}^e_{in}}{j\omega\varepsilon_0} - j\omega\mu_0[\tilde{\varepsilon}^{-1}\nabla\times(\tilde{\varepsilon}\tilde{\mu}\vec{\Pi}^m) - \nabla\times(\tilde{\mu}\vec{\Pi}^m)], \tag{22}$$

$$\left[k_0^2\tilde{\varepsilon}+\nabla\otimes\nabla\tilde{\mu}^{-1}-\tilde{\mu}^{-1}\nabla\times\nabla\times\right]\tilde{\mu}\vec{\Pi}^m=-\frac{\tilde{\mu}^{-1}\vec{j}_{in}^{\,m}}{j\omega\mu_0}+j\omega\varepsilon_0\left[\tilde{\mu}^{-1}\nabla\times\left(\tilde{\mu}\tilde{\varepsilon}\vec{\Pi}^e\right)-\nabla\times\left(\tilde{\varepsilon}\vec{\Pi}^e\right)\right]. \quad (23)$$

For nonmagnetic media the equation (23) gets the view of Helmholtz equation with the right part $-\left(j\omega\mu_0\right)^{-1}\vec{j}_{in}^{\,m}$. Then for anisotropic case the separation in the (22) occurs if $\overline{\overline{rot}}\tilde{\varepsilon}\vec{\Pi}^m=\overline{\overline{rot}}\vec{\Pi}^m$, and for isotropic case – if $\nabla\tilde{\varepsilon}\times\vec{\Pi}^m=0$, i.e. the $\nabla\tilde{\varepsilon}$ and $\vec{\Pi}^m$ are collinear. In general case nonhomogeneous and nonmagnetic media we have the equation

$$\left[k_0^2\tilde{\mu}\tilde{\varepsilon}+\nabla^2+\tilde{\varepsilon}^{-1}\left(\nabla^2\tilde{\varepsilon}-\nabla\tilde{\varepsilon}\left(\nabla\cdot\vec{\Pi}^e\right)-\nabla\tilde{\varepsilon}\times\left(\nabla\times\vec{\Pi}^e\right)\right)\right]\vec{\Pi}^e=-\frac{\tilde{\varepsilon}^{-1}\vec{j}_{in}^{\,e}}{j\omega\varepsilon_0}-j\omega\mu_0\left[\tilde{\varepsilon}^{-1}\nabla\tilde{\varepsilon}\times\left(\vec{\Pi}^m\right)\right],$$

which is simplified and separable at $\nabla\tilde{\varepsilon}\times\vec{\Pi}^m=0$. The case $\tilde{\varepsilon}=\hat{I}$ is considered analogously. Notice that instead of Hertz vectors introduction one may use with such success the vector-functions $\tilde{\varepsilon}\vec{\Pi}^e$ and $\tilde{\mu}\vec{\Pi}^m$. Moreover, we can confront the above introduced potentials with corresponding Hertz vectors. Here we introduce yet the following vector-potentials:

$$\vec{E}=\left(\nabla\otimes\nabla+k_0^2\right)\vec{\vec{\Pi}}^e-j\omega\mu_0\nabla\times\vec{\vec{\Pi}}^m \quad , \quad (24)$$

$$\vec{H}=\left(\nabla\otimes\nabla+k_0^2\right)\vec{\vec{\Pi}}^m+j\omega\varepsilon_0\nabla\times\vec{\vec{\Pi}}^e \quad , \quad (25)$$

$$\left[k_0^2+\nabla\otimes\nabla-\tilde{\varepsilon}^{-1}\nabla\times\nabla\times\right]\vec{\vec{\Pi}}^e=-\frac{\tilde{\varepsilon}^{-1}\vec{j}_{in}^{\,e}}{j\omega\varepsilon_0}-j\omega\mu_0\left(\tilde{\varepsilon}^{-1}-\hat{I}\right)\nabla\times\vec{\vec{\Pi}}^m \quad , \quad (26)$$

$$\left[k_0^2+\nabla\otimes\nabla-\tilde{\mu}^{-1}\nabla\times\nabla\times\right]\vec{\vec{\Pi}}^m=-\frac{\tilde{\mu}^{-1}\vec{j}_{in}^{\,m}}{j\omega\mu_0}+j\omega\varepsilon_0\left(\tilde{\mu}^{-1}-\hat{I}\right)\nabla\times\vec{\vec{\Pi}}^e. \quad (27)$$

Depending on specific problem the choice of different potentials may be useful and convenient. In the view of equations separation the mentioned choice less substantial that as for receiving of solutions separated equations. These equations have different forms for different potentials, and some of them may be more founded and preferable. The general approach to problem solution for the separable equations should be concluded in there Green's functions determination. These GFs must in general case be tensors as define the vector values (potentials) by vector current densities (which may be multiplied on a certain matrix). The mentioned GFs at the presence of boundaries even for homogeneous and isotropic medium, as their components must satisfy different boundary conditions [34]. When the potentials are substituting into the field expressions, one can get the others GFs which are directly connecting the fields with its sources – the current densities. Let examine, for example, nonmagnetic case of potentials (8), (9). Then, if the magnetic vector-potential is collinear to the gradient of permittivity, it is n necessary to find the GF satisfied the equation

$$\left[\nabla^2+k_0^2\tilde{\varepsilon}\right]\hat{G}^e\left(\vec{r},\vec{r}\,'\right)+\left(\tilde{\varepsilon}\nabla\tilde{\varepsilon}^{-1}\right)\otimes\left(\nabla\cdot\hat{G}^e\left(\vec{r},\vec{r}\,'\right)\right)=\hat{I}\delta\left(\vec{r}-\vec{r}\,'\right). \quad (28)$$

Let one has the particular case $\tilde{\varepsilon}\left(\vec{r}\right)=\tilde{\varepsilon}\left(x\right)$, $\vec{j}_{in}^{\,e}\left(\vec{r}\right)=\vec{x}_0J\left(\vec{r}\right)$, $\vec{\vec{A}}^e\left(\vec{r}\right)=\vec{x}_0\Phi\left(\vec{r}\right)$. Then the problem is simplified: it is necessary to solve the equation

$$\left[\nabla^2+k_0^2\right]\Phi\left(\vec{r}\right)+k_0^2\left(\tilde{\varepsilon}\left(x\right)-1\right)\Phi\left(\vec{r}\right)-\frac{1}{\tilde{\varepsilon}\left(x\right)}\frac{d\tilde{\varepsilon}\left(x\right)}{dx}\frac{\partial}{\partial x}\Phi\left(\vec{r}\right)=-J\left(\vec{r}\right). \quad (29)$$

Expressing the introduced functions via their spatial spectra

$$\Phi\left(\vec{r}\right)=\frac{1}{\left(2\pi\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi\left(\vec{k}\right)\exp\left(-j\vec{k}\vec{r}\right)d^3k\,,$$

$$\delta\varepsilon\left(x\right)=\tilde{\varepsilon}\left(x\right)-1=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta\varepsilon\left(k_x\right)\exp\left(-jk_xx\right)dk_x\,,$$

$$\varepsilon'(x)=\frac{1}{\widetilde{\varepsilon}(x)}\frac{d\widetilde{\varepsilon}(x)}{dx}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon'(k_x)\exp(-jk_x x)dk_x\ ,$$

one gets the integral equation (IE) for the desired spectral function $\Phi(\vec{k})$:

$$\Phi(\vec{k})\left[\vec{k}^2-k_0^2\right]-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[k_0^2\delta\varepsilon(k_x)\Phi(k_x-k_x',k_y,k_z)+j(k_x-k_x')\varepsilon'(k_x)\Phi(k_x-k_x',k_y,k_z)\right]dk_x'=J(\vec{k}). \tag{30}$$

It is not very convenient to solve the equations like (30) in spectral region, in which connection then these solutions would be Fourier transformed and differentiated in order to get the fields. Besides the similar equations are not general. It is easy to find the solution of equation (10) with operator in the left part which corresponds to equation (28). We use for this the scalar GF and turn into integrodifferential equation (IDE)

$$\vec{\widetilde{A}}^e(\vec{r})=\int_V G(k_0,\vec{r}-\vec{r}')\left[\vec{j}_{in}^{\,e}(\vec{r}')+k_0^2(\widetilde{\varepsilon}(\vec{r}')-1)\vec{\widetilde{A}}^e(\vec{r}')-\widetilde{\varepsilon}^{-1}(\vec{r}')\nabla'\widetilde{\varepsilon}(\vec{r}')\nabla'\cdot\vec{\widetilde{A}}^e(\vec{r}')\right]d^3r'\ . \tag{31}$$

To solve this, we calculate the fields under the relations (8), (9). The operator B (31) оператор «nabla» or inverted delta ∇' operates on primed coordinates. One may also consider the coupled IDE for the equations (10), (11). It is more convenient to get the solutions of equations (14), (15). The corresponding to them coupled IDE we write in the form:

$$\vec{\widetilde{A}}^e(\vec{r})=\int_V G(k_0,\vec{r}-\vec{r}')\widetilde{\varepsilon}^{-1}(\vec{r}')\left\{\vec{j}_{in}^{\,e}(\vec{r}')+\left(\widetilde{\varepsilon}(\vec{r}')-\hat{I}\right)\left[\nabla'\times\nabla'\times\vec{\widetilde{A}}^e(\vec{r}')-j\omega\varepsilon_0\nabla'\times\vec{\widetilde{A}}^m(\vec{r}')\right]\right\}d^3r', \tag{32}$$

$$\vec{\widetilde{A}}^m(\vec{r})=\int_V G(k_0,\vec{r}-\vec{r}')\widetilde{\mu}^{-1}(\vec{r}')\left\{\vec{j}_{in}^{\,m}(\vec{r}')+\left(\widetilde{\mu}(\vec{r}')-\hat{I}\right)\left[\nabla'\times\nabla'\times\vec{\widetilde{A}}^m(\vec{r}')+j\omega\mu_0\nabla'\times\vec{\widetilde{A}}^e(\vec{r}')\right]\right\}d^3r'. \tag{33}$$

The integration in (31) and (33) must carry out over all volume where the integrands differ from zero. Usually the incident currents are located in finite region, and the matrixes $\widetilde{\varepsilon}$, $\widetilde{\mu}$ outside the finite structures take one's stand unite. Solving the coupled IDE (32) and (33), we define the fields using the formulas (12), (13). One may write the similar coupled IDEs so many how many different potentials have been introduced. One of the such forms has been got in [18,19]. Particularly, for the equations (5) and (6) we have

$$\begin{aligned}\vec{A}^e(\vec{r})=&\int_V G(k_0,\vec{r}-\vec{r}')\left\{\vec{j}_{in}^{\,e}(\vec{r}')+\left[\widetilde{\varepsilon}(\vec{r}')\nabla'\nabla'\cdot\left(\widetilde{\varepsilon}^{-1}(\vec{r}')\vec{A}^e(\vec{r}')\right)-\nabla'\nabla'\cdot\vec{A}^e(\vec{r}')\right]+\right.\\&\left.+k_0^2\left(\widetilde{\varepsilon}(\vec{r}')\widetilde{\mu}(\vec{r}')-\hat{I}\right)\vec{A}^e(\vec{r}')+j\omega\varepsilon_0\left(\nabla'\times\widetilde{\varepsilon}(\vec{r}')-\widetilde{\varepsilon}(\vec{r}')\nabla'\times\right)\vec{A}^m(\vec{r}')\right\}d^3r',\end{aligned} \tag{34}$$

$$\begin{aligned}\vec{A}^m(\vec{r})=&\int_V G(k_0,\vec{r}-\vec{r}')\left\{\vec{j}_{in}^{\,m}(\vec{r}')+\left[\widetilde{\mu}(\vec{r}')\nabla'\nabla'\cdot\left(\widetilde{\mu}^{-1}(\vec{r}')\vec{A}^m(\vec{r}')\right)-\nabla'\nabla'\cdot\vec{A}^m(\vec{r}')\right]+\right.\\&\left.+k_0^2\left(\widetilde{\mu}(\vec{r}')\widetilde{\varepsilon}(\vec{r}')-\hat{I}\right)\vec{A}^m(\vec{r}')+j\omega\mu_0\left(\nabla'\times\widetilde{\mu}(\vec{r}')-\widetilde{\mu}(\vec{r}')\nabla'\times\right)\vec{A}^e(\vec{r}')\right\}d^3r'.\end{aligned} \tag{35}$$

Note that the volume IEs are traditionally introduced using the GF of free space $G(k_0,\vec{r})$ for presentation of vector-potentials $\vec{A}_0^{(e,m)}$ by the medium polarization current densities $\vec{J}_p^{\,e}(\vec{r})=j\omega\varepsilon_0\left(\widetilde{\varepsilon}(\vec{r})-\hat{I}\right)\vec{E}(\vec{r})$, $\vec{J}_p^{\,m}(\vec{r})=j\omega\mu_0\left(\widetilde{\mu}(\vec{r})-\hat{I}\right)\vec{H}(\vec{r})$:

$$\vec{A}_0^{(e,m)}(\vec{r})=\int_V G(k_0,\vec{r}-\vec{r}')\left[\vec{j}_{in}^{\,(e,m)}(\vec{r}')+\vec{J}_p^{\,(e,m)}(\vec{r}')\right]d^3r', \tag{36}$$

that corresponds to formal extraction of mentioned densities in Maxwell equations and they addition to incident current densities. The potentials (36) satisfy the equations (7) under the unit permittivity and permeability and with the right parts as $-\vec{j}^{\,(e,m)}-\vec{J}_p^{\,(e,m)}$. The differentiation of (36) according to formulas (16), (17) with unit material parameters leads

to volume IEs for the fields. These IEs contain the hypersingular singularities, and the corresponding operators also are named pseudo-differential [35,36]. There are several various equivalent forms of IEs and IDEs for the field, which may be obtained both using the representation (36), or by other ways. They, in particular, may contain the terms with the surface integrals.

The usability of IDEs for potentials is concluded in absence of surface integral terms and hypersingularities. They contain also one kind of integrable singularity $\left|\vec{r}-\vec{r}'\right|^{-1}$, and in this time there is the necessity of its differentiation in order to get the fields. The solutions of equations (31), (32)–(33), (34) – (35) and similar to them we will base on the potential decompositions over full systems of vector-basis functions. As the example let consider the problem (34), (35) and the decomposition

$$\vec{A}^{(e,m)}(\vec{r}) \approx \sum_{n=1}^{N} \alpha_n^{(e,m)} \vec{u}_n^{(e,m)}(\vec{r}). \tag{37}$$

By virtue of used relations the vector-functions $\vec{u}_n^{(e,m)}(\vec{r})$ must be, at least, twice continuously differentiable functions. We will consider their second derivatives as Lipschitz-continuous. It is important to propose so continuously differentiability and boundness of tensor permittivity and permeability and the existence of their inverse values. The case of discontinuous permittivity and permeability must be separately considered matching thee solutions on the boundaries of breaks. In general consideration the breaks lead to the delta-functions extractions and their derivatives, i.e. to appearance of surface integrals. Note that the singularity $\left|\vec{r}-\vec{r}'\right|^{-1}$ in the first surface integral more strong as the similar in the volume one. If one considers the for the functions of class C^{∞}, i.e. the indefinitely differentiable ones, that on account of compactness of integral operator [37] и and boundness (or continuity) of differential operator we have that the integrodifferential operators are compact (or with the Fredholm property), i.e. the customary solvability conditions fulfill. The relations (34) and (35) at that are infinitely differentiable. Note that appropriate media have the smooth boundary bedding interface, i.e. have not the steps and the steps of derivatives. On can prove the differentiability of relations (34), (35) directly by transfer of operators $\nabla\cdot$, ∇ and $\nabla\times$ on primed coordinates (or source point) using the corresponding integral theorems about the divergence, gradient and rotor, and also the property $\nabla G(\vec{r}-\vec{r}') = -\nabla' G(\vec{r}-\vec{r}')$. The mentioned theorems proved for continuously differentiable functions are spreadable on GF $G(\vec{r}-\vec{r}')$ by the way of observations point δ – neighborhood separation and then proceeding to limit $\delta \to 0$.

It was proposed in getting of (31), (35) that the souses are located inside the region where the permittivity and permeability differ from unity and have the same extent of smoothness that the potentials. We can hold their as Lipschitz-continuous functions, and if they are located at infinity – to introduce into the equations corresponding free (or the outside the integral) terms respective to plane waves. The solutions of IDEs when the second derivatives of potentials are Lipschitz-continuous are twice differentiable and they define the fields by differentiation. Actually, let introduce the vector-functions $u = (\vec{u}^{\,e}, \vec{u}^{\,m})$:

$\|u\|^2 = \|\vec{u}^{\,e}\|^2 + \|\vec{u}^{\,m}\|^2$ norm, where

$$\|\vec{u}^{\,e}\|^2 = \int_V \left| \left[\left(\hat{I} + \tilde{\varepsilon}(\vec{r})\tilde{\mu}(\vec{r}) \right) + \nabla\nabla\cdot + \tilde{\varepsilon}(\vec{r})\nabla\nabla\cdot\tilde{\varepsilon}^{-1}(\vec{r}) + \nabla\times\tilde{\mu}(\vec{r}) + \tilde{\mu}(\vec{r})\nabla\times \right] \vec{u}^{\,e}(\vec{r}) \right|^2 d^3 r,$$

$$\|\vec{u}^{\,m}\|^2 = \int_V \left| \left[\left(\hat{I} + \tilde{\mu}(\vec{r})\tilde{\varepsilon}(\vec{r}) \right) + \nabla\nabla\cdot + \tilde{\mu}(\vec{r})\nabla\nabla\cdot\tilde{\mu}^{-1}(\vec{r}) + \nabla\times\tilde{\varepsilon}(\vec{r}) + \tilde{\varepsilon}(\vec{r})\nabla\times \right] \vec{u}^{\,m}(\vec{r}) \right|^2 d^3 r.$$

The problem (34), (35) in this Hilbert space presents itself the coupled integrodifferential Fredholm second kind equations with the kernel G. This problem is solvable in terms of

average convergence. Let show the differentiability of solutions. They constitute the expressions like

$$\vec{A}(\vec{r}) = \int_V G(\vec{r}-\vec{r}')\vec{a}(\vec{r}')d^3r',$$

where $\vec{a}(\vec{r})$ is the Lipschitz-continuous vector. Directly one see the continuity and boundedness of derivatives like rotor and divergence from the vectors $\vec{A}^{(e,m)}$ and also from these vectors multiplied на on continuously differentiable material matrixes. Let consider the $\nabla\nabla\cdot\vec{A}(\vec{r})$ and break up the volum to δ – neighborhood of point $\vec{r}$ and remaining region in which all derivatives are continuous and bounded. We write for δ – neighborhood the GF as $G(k_0,\vec{r}) = G(0,\vec{r}) + \left[G(k_0,\vec{r}) - G(0,\vec{r})\right]$, where the function $\Delta G(k_0,\vec{r}) = \left[G(k_0,\vec{r}) - G(0,\vec{r})\right]$ is regular. The derivatives from respective integrals exist and are bounded. The module of integral with the function $\Delta G(k_0,\vec{r})$ may be done less than any established ε by choice of δ. Then we have the evaluation

$$\left|\nabla\nabla\cdot\vec{A}(\vec{r}) - \int_{V-\delta}(\vec{a}(\vec{r}')\cdot\nabla)\nabla G(k_0,\vec{r}-\vec{r}')d^3r'\right| = \left|\nabla\int_\delta \vec{a}(\vec{r}')\cdot\nabla G(k_0,\vec{r}-\vec{r}')d^3r'\right| \le$$

$$\le \varepsilon + \left|\nabla\int_\delta |\vec{a}(\vec{r}')-\vec{a}(\vec{r})|\nabla G(0,\vec{r}-\vec{r}')|d^3r'\right| + \left|\nabla\int_\delta \frac{\vec{a}(\vec{r})\cdot(\vec{r}-\vec{r}')}{4\pi|\vec{r}-\vec{r}'|^3}d^3r'\right| \le$$

$$\le \varepsilon + \delta|\nabla C(\vec{r})| + \frac{\delta}{2\pi}\left|\nabla|\vec{a}(\vec{r})|\int_0^\pi \cos(\theta)\sin^2(\theta)d\theta\right| = \varepsilon + \delta|\nabla C(\vec{r})|.$$

We have used the condition $|\vec{a}(\vec{r}')-\vec{a}(\vec{r})| < C(\vec{r})|\vec{r}'-\vec{r}|$ in the receiving it, and the the integrals have been calculated in spherical coordinate system with the center in the point $\vec{r}$ and with the *z*-axis directed along the vector $\vec{a}$. Correspondingly the integrals in the expressions (3), (4) must be understood from the viewpoint of principal values. As $\nabla^2\vec{A}(\vec{r}) = -\vec{a}(\vec{r}) + k_0^2\vec{A}(\vec{r})$, that the solution is also differentiable with usage of operator $rot \bullet rot$.

When the permittivity and/or permeability are the step changed then the more convenient the IDEs (32), (33) as they do not contain the derivatives, therefore there are not any surface integrals. But for the fields such integrals may appear. Let S is closed surface bounded the volume V, and there are the steps on it: $\tilde{\varepsilon}^+ = \hat{I}$, $\tilde{\varepsilon}^- \ne \hat{I}$, $\tilde{\mu}^+ = \hat{I}$, $\tilde{\mu}^- \ne \hat{I}$. Let also the sources only are inside the volume and are described by smooth functions. Then the electric field may be presentable as $\vec{E} = \vec{E}_{in} + \vec{E}^e + \vec{E}^m$, where

$$j\omega\varepsilon_0\vec{E}^e(\vec{r}) = -\oint_S \nabla G(k_0,\vec{r}-\vec{r}')\vec{\nu}(\vec{r}')\cdot\left(\hat{I}-\tilde{\varepsilon}^{-1}(\vec{r}')\right)\left[\nabla'\times\nabla'\times\vec{\tilde{A}}^e(\vec{r}') - j\omega\varepsilon_0\nabla'\times\vec{\tilde{A}}^m(\vec{r}')\right]d^2r' -$$

$$-\oint_S \vec{\nu}(\vec{r}')G(k_0,\vec{r}-\vec{r}')\nabla'\cdot\left(\hat{I}-\tilde{\varepsilon}^{-1}(\vec{r}')\right)\left[\nabla'\times\nabla'\times\vec{\tilde{A}}^e(\vec{r}') - j\omega\varepsilon_0\nabla'\times\vec{\tilde{A}}^m(\vec{r}')\right]d^2r' +$$

$$+\int_V G(k_0,\vec{r}-\vec{r}')\left(k_0^2+\nabla'\nabla'\cdot\right)\left(\hat{I}-\tilde{\varepsilon}^{-1}(\vec{r}')\right)\left[\nabla'\times\nabla'\times\vec{\tilde{A}}^e(\vec{r}') - j\omega\varepsilon_0\nabla'\times\vec{\tilde{A}}^m(\vec{r}')\right]d^3r',$$

$$j\omega\mu_0\vec{E}^m(\vec{r}) = \oint_S \vec{\nu}(\vec{r}')\times\left\{G(k_0,\vec{r}-\vec{r}')\left(\hat{I}-\tilde{\mu}^{-1}(\vec{r}')\right)\left[\nabla'\times\nabla'\times\vec{\tilde{A}}^m(\vec{r}') + j\omega\mu_0\nabla'\times\vec{\tilde{A}}^e(\vec{r}')\right]\right\}d^2r' -$$

$$-\int_V G(k_0,\vec{r}-\vec{r}')\nabla'\times\left(\hat{I}-\tilde{\mu}^{-1}(\vec{r}')\right)\left[\nabla'\times\nabla'\times\vec{\tilde{A}}^m(\vec{r}') + j\omega\mu_0\nabla'\times\vec{\tilde{A}}^e(\vec{r}')\right]d^3r',$$

$$\vec{E}_{in}(\vec{r}) = \int_V G(k_0, \vec{r} - \vec{r}') \left\{ \frac{k_0^2 + \nabla'\nabla' \cdot}{j\omega\varepsilon_0} \tilde{\varepsilon}^{-1}(\vec{r}') \vec{j}_{in}^{\,e}(\vec{r}') - \nabla' \times \tilde{\mu}^{-1}(\vec{r}') \vec{j}_{in}^{\,m}(\vec{r}') \right\} d^3 r'.$$

In order that the delta-functions do not arise in the volume integrals and do not produce the surface integral terms the potentials must have the derivatives of forth order. Getting the approximate solutions of IDEs by the method of volumetric finite elements (FE) in form(37), it is necessary subject these elements to such conditions of smoothness.

2. Potentials in bianisotropic media

The potentials in homogeneous biisotropic media were considered in the paper [9], and for bianisotropic ones – in [17]. In general case of inhomogeneous bianisotropic media the following material conditions take place [38]

$$\vec{D} = \varepsilon_0 \tilde{\varepsilon} \vec{E} + c^{-1} \hat{\xi} \vec{H}, \qquad \vec{B} = \mu_0 \tilde{\mu} \vec{H} + c^{-1} \hat{\varsigma} \vec{E}. \tag{38}$$

Introducing the vector-potentials by usual way by the electric and magnetic inductions (these potentials coincide with (3) and (4))

$$\vec{B} = \nabla \times \vec{A}^e + \frac{k_0^2 \tilde{\varepsilon}\tilde{\mu} + \nabla\nabla \cdot}{j\omega\varepsilon_0} \vec{A}^m, \qquad \vec{D} = -\nabla \times \vec{A}^m + \frac{k_0^2 \tilde{\mu}\tilde{\varepsilon} + \nabla\nabla \cdot}{j\omega\mu_0} \vec{A}^e, \tag{39}$$

taking into account the relation (38), one may expresses the field, for example

$$\vec{E} = c\varepsilon_0^{-1} \left(\tilde{\mu}^{-1} \hat{\varsigma} - \hat{\xi}^{-1} \tilde{\varepsilon} \right)^{-1} \left\{ \left[\tilde{\mu}^{-1} \frac{k_0^2 \tilde{\varepsilon}\tilde{\mu} + \nabla\nabla \cdot}{j\omega} + c^{-1} \hat{\xi}^{-1} \nabla \times \right] \vec{A}^m - \left[c\varepsilon_0 \hat{\xi}^{-1} \frac{k_0^2 \tilde{\mu}\tilde{\varepsilon} + \nabla\nabla \cdot}{j\omega} - \varepsilon_0 \tilde{\mu}^{-1} \nabla \times \right] \right\} \vec{A}^e.$$

In a similar expressing the magnetic field and substituting the fields into Maxwell equations, we get the equations for potentials. They have not presented there from behind of cumbersome form. The relations (39) present the inductions as the sums of solenoidal and potential field. However one may offer the fields themselves by one of the ways (3)–(4), (8)–(9), (12)–(13) in form of such sums. Each of presentation leads to their wave equations for potentials, that is connected with its ambiguity. As there are the four material tensors, it is more suitable to use the presentations (12), (13) independent from them. Then we obtain the equations

$$\begin{aligned}
&\left(\nabla^2 + k_0^2 \tilde{\varepsilon}\right) \vec{\tilde{A}}^e + \left(\tilde{\varepsilon} - \hat{I}\right) grad \bullet div \vec{\tilde{A}}^e + j\omega c^{-1} \hat{\xi} \nabla \times \vec{\tilde{A}}^e = \\
&= -\vec{j}_{in}^{\,e} + j\omega\varepsilon_0 \left(\tilde{\varepsilon} - \hat{I}\right) \nabla \times \vec{\tilde{A}}^m - c^{-1} \hat{\xi} \frac{k_0^2 + grad \bullet div}{\mu_0} \vec{\tilde{A}}^m, \\
&\left(\nabla^2 + k_0^2 \tilde{\mu}\right) \vec{\tilde{A}}^m + \left(\tilde{\mu} - \hat{I}\right) grad \bullet div \vec{\tilde{A}}^m - j\omega c^{-1} \hat{\varsigma} \nabla \times \vec{\tilde{A}}^m = \\
&= -\vec{j}_{in}^{\,m} - j\omega\mu_0 \left(\tilde{\mu} - \hat{I}\right) \nabla \times \vec{\tilde{A}}^e - c^{-1} \hat{\varsigma} \frac{k_0^2 + grad \bullet div}{\varepsilon_0} \vec{\tilde{A}}^e
\end{aligned}.$$

Other important problem at the boundary problem solution besides the potentials choose is the usage of minimal number of independent components (scalar functions) and its foundation. The maximal number such components is six as the number of field components. But not always they are independent. It is sufficient for electric ot magnetic dipole in free space one scalar function. In case of simultaneously electrical and magnetic excitations there are two ones [1-9,11-25]. If there are the inhomogeneous and anisotropic dielectric media it is necessary four scalar components or potentials (as for only magnetic media). If there are the boundary bedding interface and the screens (or region boundaries) then the potentials in subregions are used. The fields at the boundaries must be matched by conjugation relations or boundary conditions. The planar parallel surfaces and boundaries are more considerable in literature. For these, if the filling is homogeneous, also it is sufficient of two scalar components. It already is not implemented for inhomogeneous

media. The particular case with two potentials in inhomogeneous dielectric is given in [13]. If the partial region is the generalized parallelepiped in orthogonal curvilinear coordinates, then it is necessary to use the six such components [39]. particularly, two GFs of rectangular waveguide connected the electric and magnetic vector-potentials with the currents are the diagonal tensors, Its components describe the reactions of potentials on the currents of corresponding dipoles (electric and magnetic), which are oriented along each of the axis [39]. This also corresponds to inhomogeneous and anisotropic media. In general case it need to introduce two scalar potential components $\Phi^{e}_{x_I}$ and $\Phi^{m}_{x_I}$ for each boundary or aggregate boundaries in each partial subregion, which are determined by the equation $x_I = const$ in generalized coordinates.

3. Numerical results

Let consider the open isotropic cylindrical dielectric ($\widetilde{\mu} = \hat{I}$) resonator (CDR) with scalar permittivity having the distribution $\widetilde{\varepsilon}(\vec{r}) = 1$, if $\rho > \rho_0$, $|z| > h/2$ and

$$\widetilde{\varepsilon}(\vec{r}) = \varepsilon_\rho(\rho)\varepsilon_z(z) = \left[1 + \kappa_1 \cos^m\left(\frac{\pi\rho}{2\rho_0}\right)\right]\left[1 + \kappa_2 \cos^n\left(\frac{\pi z}{h}\right)\right], \quad \rho \le \rho_0, \quad |z| \le h/2, \tag{40}$$

where the ρ, φ, z are the cylindrical coordinates. Here ρ_0 is the radius of resonator, h is its height, m, n are the integer numbers. In order that the function (40) should be twice continuously differentiable it is necessary to take $m, n \ge 3$, i.e. to consider the body with smooth boundary. We will analyze the characteristic oscillation. Then it is follow from (32), (33) that $\vec{\vec{A}}^m(\vec{r}) = 0$,

$$\vec{\vec{A}}^e(\vec{r}) = \int_V G(k_0, \vec{r} - \vec{r}')\left(\hat{I} - \widetilde{\varepsilon}^{-1}(\vec{r}')\right)\nabla' \times \nabla' \times \vec{\vec{A}}^e(\vec{r}')d^3r' . \tag{41}$$

The potential of azimuthally symmetric oscillations is the function of ρ and z. In general case we have the three potential components A_ρ, A_φ, A_z. However, if $A_\varphi = 0$ then the magnetic field has unique component $H_\varphi(\rho, z)$, and the equation (41) for two independent components A_ρ, A_z obtains the form of coupled scalar IDEs:

$$A_\rho(\rho, z) = \int_0^{\rho_0}\int_{-h/2}^{h/2} g(k_0; \rho, z \mid \rho', z')\left(1 - \widetilde{\varepsilon}^{-1}(\rho', z')\right)\left[\frac{\partial^2 A_z(\rho', z')}{\partial\rho'\partial z'} - \frac{\partial^2 A_\rho(\rho', z')}{\partial z'^2}\right]\rho' d\rho' d\varphi' dz' , \tag{42}$$

$$\begin{aligned} A_z(\rho, z) &= \int_0^{\rho_0}\int_{-h/2}^{h/2} g(k_0; \rho, z \mid \rho', z')\left(1 - \widetilde{\varepsilon}^{-1}(\rho', z')\right)\cdot \\ &\cdot\left[\left(1 + \rho'\frac{\partial}{\partial\rho'}\right)\frac{\partial A_\rho(\rho', z')}{\partial z'} - \left(\frac{\partial}{\partial\rho'} + \rho'\frac{\partial^2}{\partial\rho'^2}\right)A_z(\rho', z')\right]d\rho' d\varphi' dz'. \end{aligned} \tag{43}$$

Here the kernel has the following presentations [28,40]:

$$g(k_0; \rho, z \mid \rho', z') = \int_0^{2\pi} G(k_0, \vec{r} - \vec{r}')\cos(\varphi - \varphi')d\varphi' =$$

$$= 2\int_0^{\pi} G(k_0, \vec{r} - \vec{r}')\cos(\varphi - \varphi')d\varphi' =$$

$$= \frac{-j}{4}\int_{-\infty}^{\infty}\exp(-j\gamma(z - z'))\begin{Bmatrix} H_1^{(2)}(\kappa\rho)J_1(\kappa\rho') \\ J_1(\kappa\rho)H_1^{(2)}(\kappa\rho') \end{Bmatrix}d\gamma =$$

$$= \frac{-j}{4}\int_0^{\infty} \exp\left(-j\sqrt{k_0^2-\chi^2}\,|z-z'|\right)\begin{Bmatrix} H_1^{(2)}(\chi\rho)J_1(\chi\rho') \\ J_1(\chi\rho)H_1^{(2)}(\chi\rho') \end{Bmatrix} \frac{\chi d\chi}{\sqrt{k_0^2-\chi^2}} =$$

$$= \frac{-j}{4}\int_0^{\infty} \exp\left(-j\sqrt{k_0^2-\chi^2}\,|z-z'|\right) \frac{J_1(\chi\rho)J_1(\chi\rho')\chi d\chi}{\sqrt{k_0^2-\chi^2}}.$$

Here $\kappa = \sqrt{k_0^2-\gamma^2}$, $\mathrm{Im}(\kappa) \le 0$, the upper values of functions in figure brackets are taken at $\rho > \rho'$, and the lower ones – at $\rho < \rho'$. If we have one potential component $A_\varphi(\rho,z)$ then there are the oscillations with one electric field component E_φ. The potential in this case satisfy the equation

$$A_\varphi(\rho,z) = -\int_0^{\rho_0}\int_{-h/2}^{h/2} g(k_0;\rho,z\,|\,\rho',z')\left(1-\tilde{\varepsilon}^{-1}(\rho',z')\right)\left[\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)+\rho'\frac{\partial^2}{\partial z'^2}\right]A_\varphi(\rho',z')d\rho' dz', \tag{44}$$

which still may be presentable in two forms:

$$A_\varphi(\rho,z) = k_0^2\int_0^{\rho_0}\int_{-h/2}^{h/2} g(k_0;\rho,z\,|\,\rho',z')\left(\tilde{\varepsilon}(\rho',z')-1\right)A_\varphi(\rho',z')\rho' d\rho' dz', \tag{45}$$

$$A_\varphi(\rho,z) = k_0^2\int_0^{\infty}\int_{-\infty}^{\infty} \tilde{\varepsilon}(\rho',z')\,g(0;\rho,z\,|\,\rho',z')A_\varphi(\rho',z')\rho' d\rho' dz'. \tag{46}$$

The equation (45) coincides with the IE for E_φ, which is describing the $H_{0m\delta}$- modes of CDR [40], because $E_\varphi = -j\omega\mu_0 A_\varphi$. The equation (46) is considering in infinite region with the solution behavior in far zone as $A_\varphi(\rho,z) \approx A_{0\varphi}(\rho,z) = A_0\exp\left(-jk_0\sqrt{\rho^2+z^2}\right)/\sqrt{\rho^2+z^2}$. The values A_φ and k_0 are complex and $\mathrm{Im}(k_0) > 0$ that means the potential increase with the growth of distance from DR. The constant A_0 may be defined using the known potential distribution inside the DR. Since there are not the material parameter derivatives in considered IDEs we may use also the discontinuous distributions. The spectral parameter $\lambda = 1/k_0^2$ comes into the IE (46) nonlinearly.

Let consider the iteration methods of equations (42)–(45) solution writing they in operator form $A = \hat{L}(k_0)A$. Here $\hat{L}$ is the linear integrodifferential or integral operator, and A defines the totality of components, for example $A = (A_\rho, A_z)$ for (42), (43). Here we use the vector-function components decompositions (37) by one-dimensional (1-D) FEs $u_n(x)$: $u_n(x) = u_0(x-n\Delta x)$, $n = 0,\pm1,\pm2,...$, determined on three nodes where

$$u_0(x) = \begin{cases} \left[1-(x/\Delta x)^2\right]^l, & |x| \le \Delta x, \\ 0, & |x| > \Delta x. \end{cases} \tag{47}$$

These FEs are set on the nodes of uniform 1-D grid with the step Δz. When $l = 0$ we have the piecewise constant FEs. When $l = 1$ the FEs are continuous and differentiable inside the region of definition: $w_n = u_n'(x) = -2\Delta x^{-2}(x-n\Delta x)$. The derivative has the jumps at the boundary from zero up to $2/\Delta x$ at the left and from $-2/\Delta x$ to zero from the right. The second derivative in the region of bearer is constant and equal $-2/\Delta x^2$, and outside the region it is equal to zero, i.e. is the piecewise constant. If $l = 2$ then the first derivative is continuous, and if $l = 3$ then the second one is continuous. These FEs are biorthogonal. But if $l \ge 2$ then the first derivative in the nodes is equal to zero that describes its

decomposition not very satisfactorily. It is more useful to extend the bearer in (47), for example up to five nodes.

We will consider the even and odd regarding to z oscillations dividing the regions области $(0,\rho_0)$ and $(0,h/2)$ correspondingly on $M+1$ and on $N+1$ nodes by the intervals $\Delta\rho=\rho_0/M$, $\Delta z=h/(2N)$ and presenting the potentials as

$$A_\alpha(\rho,z)=\sum_{m=0}^{M}\sum_{n=0}^{N}A_{mn}^{\alpha}u_m(\rho)u_n(z). \tag{48}$$

here α denotes the ρ, z or φ. We get the matrix equations by usage the projection approach with the same weight functions and with the weight ρ for coordinate ρ. Let consider the minimal discrepancy method (MMH) [41] which write in operator form. The discrepancies of spectral parameter k_0 and equation at the iteration k are:

$$\Delta k_0^{(k)}=1-\frac{\left\langle A^{(k)},\hat{L}\left(k_0^{(k)}\right)A^{(k)}\right\rangle}{\left\langle A^{(k)},A^{(k)}\right\rangle},\qquad \Delta A^{(k)}=A^{(k)}-\hat{L}\left(k_0^{(k)}\right)A^{(k)}\ . \tag{49}$$

The Dirac brackets here mean the scalar product and the approaches $k_0^{(0)}$ and $A^{(0)}$ are set. We will get the approach $k+1$ using the approach k by the formulas:

$$k_0^{(k+1)}=k_0^{(k)}-\tau^{(k)}\Delta k_0^{(k)},\quad A^{(k+1)}=A^{(k)}-\eta^{(k)}\Delta A^{(k)},\ k=0,1,2,... \tag{50}$$

(with the posterior normalization). Here $\tau^{(k)}$, $\eta^{(k)}$ are the complex iteration parameters. These parameters are defined on each iteration step from the minimum of discrepancies (49):

$$\Delta k_0^{(k+1)}=\min_{\tau^{(k)}}\left|1-\frac{\left\langle A^{(k)},\hat{L}\left(k_0^{(k)}-\tau^{(k)}\Delta k_0^{(k)}\right)A^{(k)}\right\rangle}{\left\langle A^{(k)},A^{(k)}\right\rangle}\right|^2, \tag{51}$$

$$\Delta A^{(k+1)}=\min_{\eta^{(k)}}\left\|A^{(k)}-\eta^{(k)}\Delta A^{(k)}-\hat{L}\left(k_0^{(k+1)}\right)\left(A^{(k)}-\eta^{(k)}\Delta A^{(k)}\right)\right\|^2. \tag{52}$$

We have from (51) the equations $\left\langle A^{(k)},\hat{L}'\left(k_0^{(k)}-\tau^{(k)}\Delta k_0^{(k)}\right)A^{(k)}\right\rangle=0$ and $\left\langle A^{(k)},\hat{L}\left(k_0^{(k)}-\tau^{(k)}\Delta k_0^{(k)}\right)A^{(k)}\right\rangle=\left\langle A^{(k)},A^{(k)}\right\rangle$, in which the parameter $\tau^{(k)}$ comes nonlinearly. In order to get the global minimum it is necessary the second condition. We have from (52) the value

$$\eta^{(k)}=\frac{\left\langle\left(\hat{E}-\hat{L}\left(k_0^{(k+1)}\right)\right)\Delta A^{(k)},\left(\hat{E}-\hat{L}\left(k_0^{(k+1)}\right)\right)A^{(k)}\right\rangle}{\left\langle\left(\hat{E}-\hat{L}\left(k_0^{(k+1)}\right)\right)\Delta A^{(k)},\left(\hat{E}-\hat{L}\left(k_0^{(k+1)}\right)\right)\Delta A^{(k)}\right\rangle}, \tag{53}$$

where $\hat{E}$ is the unite operator. If the discrepancies are small one can get after the linearization of (51) the expression

$$\tau^{(k)}=\frac{\left\langle A^{(k)},\hat{L}\left(k_0^{(k)}\right)A^{(k)}\right\rangle-1}{\Delta k_0^{(k)}\left\langle A^{(k)},\hat{L}'\left(k_0^{(k)}\right)A^{(k)}\right\rangle}. \tag{54}$$

The stroke means the operator derivative on the parameter. The resulted operator does not contain the singularities. The expression (54) may be defined more precisely by usage of high order decompositions on discrepancy $\Delta k_0^{(k)}$. If the parameters η, τ are constant at each iteration step, that the algorithm of direct iteration is realized corresponding at $\eta=\tau=1$ to serial approach method (SAM). On account of not self-conjugate operator the eigenvalues of wave number k_0 are complex, therefore the resonator excitation problem is unambiguously solvable. The more laborious process is the search of complex roots of

characteristic equation for k_0 .The IE (46) is also formulated difference potential $A'_\varphi = A_\varphi - A_{0\varphi}$ in the finite regions covered the DR by introduction of discrepancies

$$\Delta\lambda^{(k)} = \lambda^{(k)} - \left\langle A'^{(k)}, \widetilde{L}A'^{(k)} \right\rangle / \left\langle A'^{(k)}, A'^{(k)} \right\rangle, \qquad \Delta A'^{(k)} = \lambda^{(k)} A'^{(k)} - \widetilde{L}A'^{(k)}.$$

Here the operator $\widetilde{L}$ doe not depend on spectral parameter $\lambda = 1/k_0^2$, and the iterations have the form

$$\lambda^{(k+1)} = \lambda^{(k)} - \tau^{(k)}\Delta\lambda^{(k)} \ , \qquad \tau^{(k)} = \left[\lambda^{(k)} - \left\langle A'^{(k)}, \widetilde{L}A'^{(k)} \right\rangle / \left\langle A'^{(k)}, A'^{(k)} \right\rangle\right] / \Delta\lambda^{(k)}.$$

This approach demands to define the constant $A_0^{(k)}$ at any step that is not inconveniently. The problem (49)-(52) is also may be reformulated for eigen values of parameter λ. One may "freeze" the operator dependence from this parameter in MDM (i.e. using the linearized process), and take it into account in the discrepancy calculation. Other algorithm may be founded on IE (45) kernel decomposition on parameter k_0 accurate within three terms: $A \approx k_0^2 \hat{L}_0 A - jk_0^3 \hat{L}_1 A - k_0^4 \hat{L}_2 A$. Here ь $\hat{L}_0 = \hat{L}(0)$, $\hat{L}_2 = \hat{L}''(0)$, and the operator $\hat{L}_1$ is zero. For the iteration *k* we have

$$\lambda^{(k)} = \delta_k = \frac{\left\langle A^{(k)}, \hat{L}_0 A^{(k)} \right\rangle + \sqrt{\left\langle A^{(k)}, \hat{L}_0 A^{(k)} \right\rangle^2 - 4\left\langle A^{(k)}, A^{(k)} \right\rangle \left\langle A^{(k)}, \hat{L}_2 A^{(k)} \right\rangle}}{2\left\langle A^{(k)}, A^{(k)} \right\rangle} \ . \qquad (55)$$

If $\widetilde{\varepsilon} >> 1$ then the equation (55) defines the real eigenfrequencies. In order to fine the radiation quality factor it si necessary to take into account the next terms of decomposition [40]. We can use also the iteration on λ with the parameter $\tau^{(k)} = \left(\lambda^{(k)} - \delta_k\right) / \Delta\lambda^{(k)}$ defined by discrepancy minimization. The processes like (50) may alternate (as it is recorded), or fulfill each several times up to good convergence at established *k*.

Eigenfrequencies f (GHz) and quality factor Q at different sizes of CDR and mode $H_{01\delta}$

DR configuration, mm	SAM, $m=0,\ n=0$		MDM, $m=1,\ n=0$		MDM, $m=1,\ n=1$		Results of [40], f $m=n=0$
	f	Q	f	Q	f	Q	
$h=7.0,\ \rho_0=5.0$	4.545879	45.46	6.530581	27.85	6.007415	36.09	4.645
$h=8.4,\ \rho_0=7.5$	3.186053	44.27	4.550707	28.01	4.225319	34.62	3.251
$h=7.0,\ \rho_0=7.5$	3.333408	42.47	4.737466	27.65	4.431561	32.94	3.383
$h=4.0,\ \rho_0=5.0$	5.210738	40.44	7.373461	27.02	6.939240	31.23	5.289

The resonance frequencies of $H_{01\delta}$ oscillations are presented in the table. They have been obtained using the IE (45) by SAM and MDM applications. Also here are the results from [40]. We have used the 36 basis functions like (47) (six nodes for each coordinate) in the region $z > 0$ with parameter $l = 1$. It is equivalent in accuracy to ten piecewise constant fragmentations. The matrix elements have been calculated numerically with the extraction of singularities. The function (40) at $\kappa_1 = 37$, $\kappa_2 = 0$ with the number $m = 0$ (homogeneous dielectric) and $m = 1$ (inhomogeneously dependent on ρ dielectric), and also at $\kappa_1 = \kappa_2 = 6.1644$, $m = n = 1$ (inhomogeneous dielectric with respect to ρ and *z*) has been used as permittivity. The potential initial approximation was set from the physical reasons as the sine and cosine product. It must comment that the results for resonant frequencies have been obtained with small (about 0.5%) lack, as they some increase with the increasing of basis functions number. Thus, the calculation in the first column of fourth line in the table at the number 16×16 of piecewise constant fragmentations leads to the

value $f = 5.216752$ GHz. The adjustment by the Eitken process gives $f = 5.231598$ GHz. On the other hand, the results [40] have been obtained, apparently, with excess, i.e. the eigen problem for static matrix have been solved, and then the asymptotic perturbation method has been applied. The iterations for such linear eigenvalue problem give the vary closed to [40] results. The convergent was achieved over several (5–7) iterations. The GF calculation was implemented over the angles by the mean-value method with usage of 60 points in the region $(0, \pi)$.

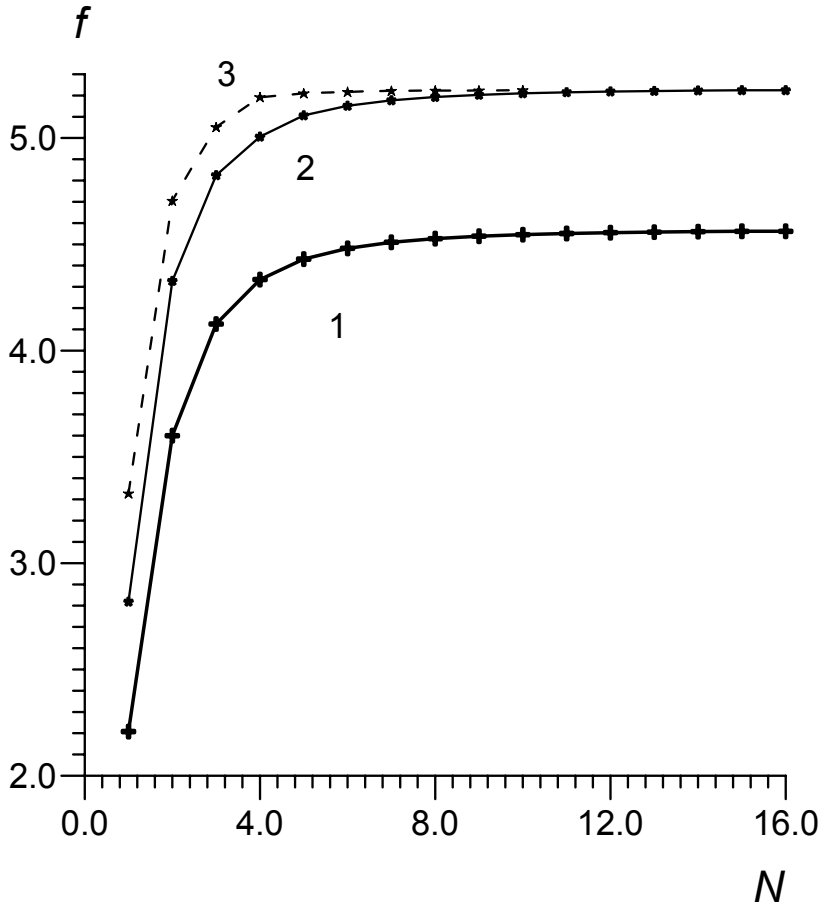

Fig. 1. The convergence of results for CDR eigenfrequencies f (GHz) from the number N of basis functions at the piecewise constant (1,2) and quadratic polynomial (3) approximations and $\varepsilon = 38$, $r_0 = 5$ mm: $h = 7$ mm (1); $h = 4$ mm (2,3)

The fig.1-5 present the results of eigenproblem solutions for CDR which are based on field IE. These results coincide with the similar one obtained by IEs vector-potentials. Also the cubic DR with homogeneous and inhomogeneous dielectric has been investigated using the proposed equations and methods. The methods are very effective and precise numerically as compared with the finite difference, traditional finite element approaches.

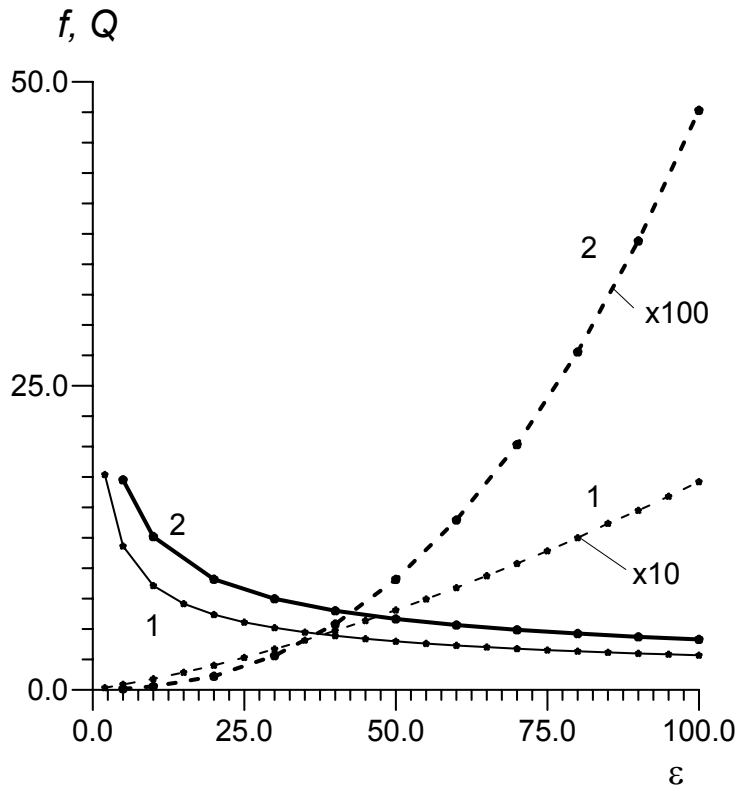

Fig. 2. The resonant frequency f (GHz) and the quality factor Q dependences for homogeneous CDR $r_0 = 5$, $h = 7$ mm versus the permittivity: 1 – mode $H_{01\delta}$, 2 – mode H_{011}

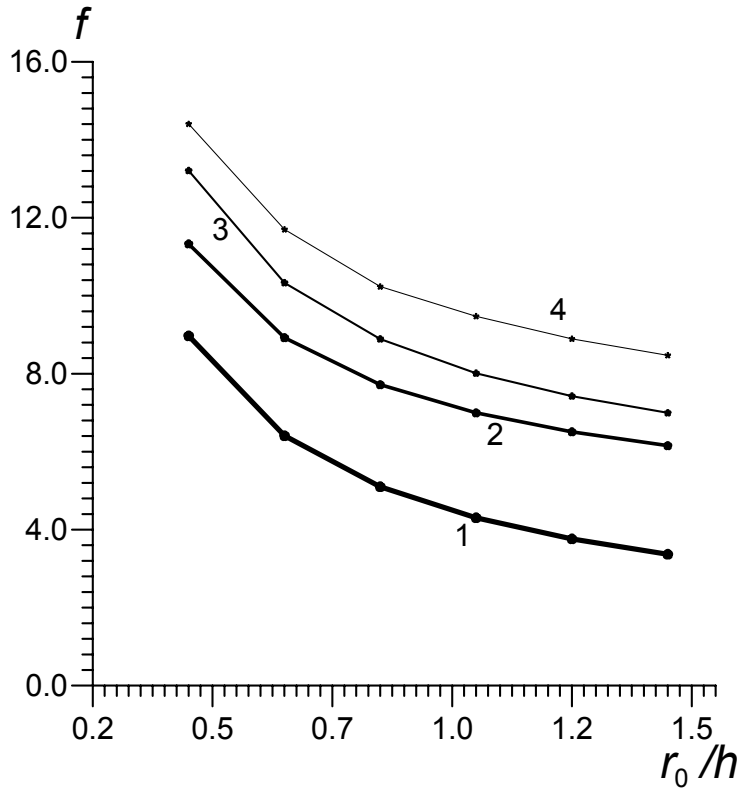

Fig. 3. The resonant frequency f (GHz) Q dependence on CDR shape for homogeneous (1,2) dielectric with $\varepsilon = 50$ and for inhomogeneous along z-axis dielectric (3,4) for $h = 5$ mm: 1 – mode $H_{01\delta}$; 2,3,4 – mode H_{011}

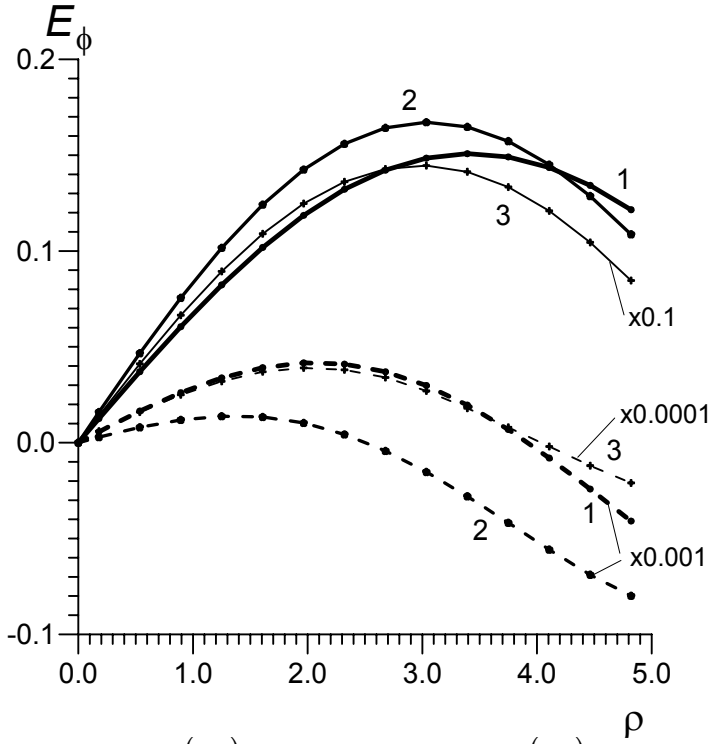

Fig. 4. The dependences $\mathrm{Re}(E_\varphi)$ (solid curves) and $\mathrm{Im}(E_\varphi)$ (dashed curves) versus the coordinate ρ (cm) for CDR $\varepsilon = 100$, $r_0 = h = 5$ mm: 1 – mode $H_{01\delta}$ at $z = 0.09$; 2 – mode H_{011} at $z = 2.41$; 3 – mode H_{011} at $z = 0.09$

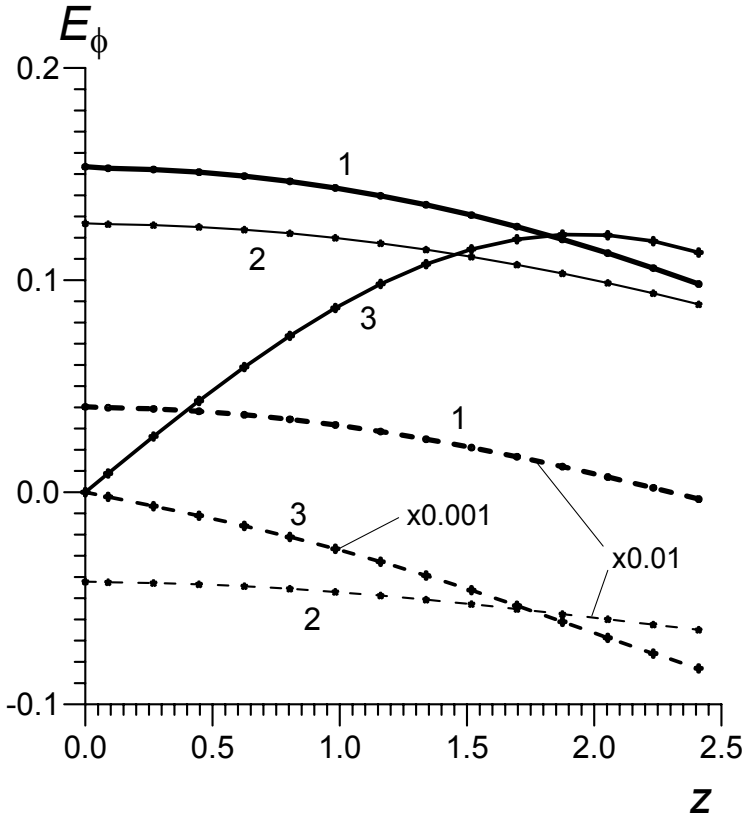

Fig. 5. The dependences $\mathrm{Re}(E_\varphi)$ (solid curves) and $\mathrm{Im}(E_\varphi)$ (dashed curves) versus z (mm) for CDR $\varepsilon = 100$, $r_0 = h = 5$ mm: 1,2 – mode $H_{01\delta}$ at $\rho = 3.4$ and $\rho = 1.66$ mm; 2 – mode H_{011} at $\rho = 4.46$ mm

Conclusions

Several possible approaches of vector electric and magnetic paired potentials introductions have been considered and obtained for anisotropic and bianisotropic media. The new general differential and integrodifferential equations have been formulated for them. The method is based on the secondary source concept [28], which means the equivalence of Maxwell equations with medium polarization currents as initial sources in vacuum to corresponding equations in medium. Conformably the all introduces fields satisfy Maxwell equations in medium. All fields and potentials as the solutions of IDEs also satisfy the boundary conditions [42,43]. For open exterior problem they satisfy radiation conditions. For closed or interior problems (for example, waveguides, cavity resonator with medium) it is necessary instead of scalar GF to use the tensor GFs which determine the potentials by way of sources. In this time thee fields are expressed trough four others tensor GFs [44]. The mentioned potential GRs for hollow rectangular waveguide and resonator ate the diagonal tensors [34] and characterize the excitation of hollow structures by dipoles oriented correspondingly along the x, or y, or z axis. The GFs for fields in this case are non-diagonal by now. The formulation volume-surface IDEs for impedance surfaces presents no difficulties. The impedance conditions applying for relations like (3) and (4) give additional equation for surface current density. It must be notice that the surface current is the low frequency idealizations. More definitely one must describe the metallic bodies as collision plasma, and in that sense the volume formulations have the general character. The efficiency of potential boundary problem formulations in comparison with field formulations has been shown. Namely, the IDEs have the weekly-singular kernels, and the potentials themselves are continuous on the media bedding interface (in contrast to fields). Furthermore, one usually seeks not the fields, but some characteristics expressed from them (the resonance frequencies, reflection and transmission coefficients, radiation patterns etc.), which can be defined by the potentials omitting the fields. The FEs and iteration algorithms have been proposed for IDEs and IEs solutions. The modes $H_{01\delta}$ are investigated numerically for homogeneous and nonhomogeneous CDRs. The potential homogeneous CDR problem here coincides with the field formulation. The radiation character here in homogeneous case is the magneto-quadrupole [40], and for the inhomogeneous filling there is the electrical dipole term yet. It must be notice that one can use the GFs of filled by medium structure. Such approach is not constructive as it not leads to the closed form of GF definition.
Let once more stress that the potential introduction in electrodynamic of continuous media is ambiguously determined. In particular, there are the works [45], [46] where such introduction is based on the relations like (1) with the additional conditions (the superfluous potentials method). The present approach in our case of arbitrary bianisotropic media has the greater generality.

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ON CONSERVATION LAWS FOR ELECTROMAGNETIC FIELD ENERGY AND MOMENTUM IN MEDIA AND FOR PLANE WAVE DIFFRACTION ON CONDUCTING MEDIUM PLATE

M.V. Davidovich, *Senior member IEEE*

Saratov State University, 410012 Saratov, Russia
E-mail: DavidovichMV@info.sgu.ru

Abstract – The general nonstationary balance equations for energy and momentum densities of field-matter system based on rigorous nonstationary approach for their definitions with dependence from the field creation prehistory have been obtained. Also the transport velocities of these densities have been derived. The concrete examination and detailed consideration have been performed for simplest dispersion law which is defined by the conductivity connected with the dissipation. There are following parameters which have been found for plane monochromatic wave under this law: the energy density, the phase velocity, the group velocity, and the transport velocities of energy and momentum. It has been shown that the energy density has the static form in which the dielectric permittivity must be replaced by its real part, and the energy transport velocity coincides with the phase velocity. The group velocity in this case may exceed the light velocity in the vacuum. It has been also shown that correct form of momentum density is the Minkowski one, and the momentum transport velocity in this case also coincides with phase velocity. The energy and momentum conservation have been shown for plane electromagnetic wave in the conducting medium and for plane wave diffraction on the conducting plate.

1. Introduction

Up to present day there is paradoxical state in the electrodynamics of continuum, when more than hundred years the energy-momentum tensor (EMT) has not got the distinctness in correct definition [1–13]. There are two principal definitions: the Minkowski [1] and the Abraham [2] correspondingly. Also there are many publications as pro Minkowski and con Abraham, for example, [13], so on the contrary for the Abraham and against the Minkowski (see, for example, publications [3–6], the latest surveys [11,12], the paper [13] and the literature there). Besides there are the papers which confirm that two stated are equivalent (for example, [3,12]), but, nevertheless the Abraham tensor is more preferable or correct. But the Minkowski tensor also may be used and it often is more convenient and more corresponding to continuum (media) [3]. The Minkowski tensor is considered as more faithful in some other publications. There are series of publications which contain the experimental confirmations or refutations of both definitions [12]. In particular, there are the publications on measurements of Abraham force $\vec{f}^{\,A}$ – the value, which is the makeweight to time derivative of Abraham momentum density: $\partial_t \vec{g}^{\,A}$. As the result of this one can get the derivative $\partial_t \vec{g}^{\,M}$ of Minkowski momentum density [3]. Notice, that all mentioned experiments have been created for quasi-stationary or nonstationary (pulse) processes, and the volumetric Abraham force is not equivalent to the sum of Lorentz forces which are acting on polarization currents [5]. Further (in order to avoid misunderstanding) we will understand that the electromagnetic pulse, or tandem, or wave train is the nonstationary wave, and the field impulse or momentum $\vec{G}$ is the volume integral of linear momentum field-matter density $\vec{g}$.

The mention ambiguity has caused the series of attempts to define and introduce the EMT otherwise, for example, using the microscopic electrodynamics [7,8], or by solving the matter equation of motion, or by using the Lagrange approach with Noether theorem [13]. It is believed that the EMT may be unambiguously defined only for the field-matter

system, and separately the indicated values are not ambiguously determined. Meanwhile, the ambiguity in momentum field density determination leads to uncertainty of momentum transfer velocity of field $\vec{v}_i^{(EM)}$ and matter $\vec{v}^{(M)}$, and so to uncertainty for full momentum transfer velocity of system field-matter $\vec{v}_i$. Also it leads to ambiguity of electromagnetic pressure force on matter.

In present paper we introduce new nonstationary balance equations for energy and momentum and the nonstationary definitions of their densities which depend on field creation process prehistory. On this basis we give the expressions for energy and momentum transfer velocities of system field-matter, and also the separately expressions for field and matter. To simplify the consideration the general results are concretized by the consideration of one-dimensional problems: а) for electromagnetic plane-wave in the medium with the dispersion caused by conductivity; б) for electromagnetic plane-wave diffraction on parallel-sided plate. The mentioned relations for monochromatic process allows one to construct the EMT and to defined the energy and momentum transfer velocities of field-matter which for stated dispersion law are coincided with the phase velocity.

2. The balance equation for energy and momentum

Always when any paradox arises, it is necessary to seek where the substitution of conceptions took place or some conceptions have been unlawfully applied to considered phenomena [14]. In case of EMT and energy-momentum densities in media it is concluded in incompetent nonstationary concepts substitution by stationary ones. In particular, the field energy density in continuum is defined in such a way: $u(\vec{r},t)=\left[\vec{D}(\vec{r},t)\vec{E}(\vec{r},t)+\vec{B}(\vec{r},t)\vec{H}(\vec{r},t)\right]/2$, i.e. as in the statics, that is not true [14–16]. Similarly the Abraham momentum density is defined as $\vec{g}^{A}=\vec{S}/c^2$, and the Minkowski one as $\vec{g}^{M}=\vec{D}(\vec{r},t)\times\vec{B}(\vec{r},t)=n^2\vec{S}(\vec{r},t)/c^2=n^2\vec{g}^{A}$, where $\vec{S}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$ is the Pointing vector, and $n=\sqrt{\varepsilon\mu}$ is the refractive index (or retardation ratio). Correspondingly in overwhelming number of dedicated to EMT works it has been supposed that the material conditions have the form $\vec{D}(\vec{r},t)=\varepsilon_0\varepsilon(\vec{r})\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)=\mu_0\mu(\vec{r})\vec{H}(\vec{r},t)$, i.e. as in the supposition of time (frequency) dispersion absence. That is also takes place only in static case. The presented notation corresponds to inhomogeneous media. The parameters ε and μ simply are the constants in majority numbers of works. We will consider the material conditions in Landau-Lifshitz form [17]:

$$\vec{D}(\vec{r},t)=\varepsilon_0\partial_t^{-1}\partial_{\vec{r}}^{-1}\left(\hat{\varepsilon}(\vec{r},\vec{r}',t-t')\vec{E}(\vec{r}',t')\right),\qquad \vec{B}(\vec{r},t)=\mu_0\partial_t^{-1}\partial_{\vec{r}}^{-1}\left(\hat{\mu}(\vec{r},\vec{r}',t-t')\vec{H}(\vec{r}',t')\right). \quad (1)$$

although there are others forms (for example, the Casimir one) [18]. Here we had introduce the following integral operators:

$$\partial_t^{-1}\left(f(t')\right)=F(t)=\int_0^t f(t')dt',\qquad \partial_{\vec{r}}^{-1}\left(\varphi(\vec{r},\vec{r}')\right)=\Phi(\vec{r})=\int_V\varphi(\vec{r},\vec{r}')d^3r', \quad (2)$$

where the scalar or vector functions may be under the integral, and $d^3r'=dV'$ is the volume element for source point. The lower limit in the first integral may be putted to infinity $-\infty$, and then the function must satisfy $f(t)\to 0$ under $t\to-\infty$. We consider the homogeneous in time processes for simplicity, thereby the kernels in (1) depend on time difference $t-t'$, i.e. the causality principle is implemented: $\hat{\varepsilon}(\vec{r},\vec{r}',t-t')=\hat{\mu}(\vec{r},\vec{r}',t-t')=0$ under $t'>t$. In the second integral the causality principle is again implemented because the volume is chosen from the condition $|\vec{r}-\vec{r}'|/c\le t-t'$. This principle means that the

contribution to the electric displacement and to magnetic induction are carried by the fields in the points which are located at the distance less than the light way $c(t-t')$. It is corresponds to taking into account the spatial dispersion (here always $t \geq t'$). Usually the spatial dispersion occupies lot smaller region, than above specified. The kernels in (1) in general case are the tensor and nonhomogeneous coordinate functions that corresponds to anisotropic inhomogeneous media. Let write the Maxwell equations in general form

$$\nabla \times \vec{H}(\vec{r},t) = \partial_t \vec{D}(\vec{r},t) + \vec{J}^e(\vec{r},t) \ , \quad -\nabla \times \vec{E}(\vec{r},t) = \partial_t \vec{B}(\vec{r},t) + \vec{J}^m(\vec{r},t) \ . \tag{3}$$

Here as usual $\nabla \equiv \partial_{\vec{r}}$ means the vector-differential operator. The meaning of Maxwell equations (3) is sufficiently easy: it is the full balance of currents, at that in the left their parts there are the full current densities (the electrical in the first equation and the magnetic in the second one), and in the right their parts stand the sums of corresponding initial and displacement current densities. The equations (3) are most general, as the all medium influence (in particularly, the conductivity current) have been taken into account in material conditions. The taking into account of electrical conductivity in form $\vec{J}^e_\sigma(\vec{r},t) = \sigma^e(\vec{r})\vec{E}(\vec{r},t)$ (i.e. as for direct current) is possible using the following kernel representation

$$\hat{\varepsilon}(\vec{r},\vec{r}',t-t') = \delta(t-t')\left[\hat{\varepsilon}(\vec{r},\vec{r}') + \left(\sigma^e(\vec{r})/\varepsilon_0\right)\partial_t^{-1}\right] + \hat{\kappa}^e(\vec{r},\vec{r}',t-t'),$$

where $\hat{\kappa}^e(\vec{r},\vec{r}',t-t')$ is the kernel of electrical receptivity operator, $\partial_t \partial_t^{-1} = I$ is the unit or identical operator. In simple case of only frequency dispersion such taking into account of conductivity corresponds to pole in the spectral function of dielectric permittivity at zero frequency [17]. In general case it is necessary to use the Drude's formula for frequency dependence of conductivity, which corresponds to integral operator kernel $\hat{\varepsilon}$ as for plasma. There is the incident magnetic current in the second equation (3). Although the unit magnetic charge (Dirac monopole) is not discovered yet, the introduction of $\vec{J}^m(\vec{r},t)$ is highly useful for symmetry since the incident magnetic currents may be equivalent to definite configurations of incident electrical currents. We will consider that the filed is absent before the time moments $t < 0$. Correspondingly the energy and momentum densities of field and matter (accurate within the self-energy) before $t < 0$ are equal to zero. The incident sources arise at the time $t_0 = 0$ and execute the work for the creation of field and change the energy and momentum of field and matter. Usually it is considered that the source energy is of nonelectromagnetic character that is convenient mathematically, though physically this energy often nevertheless is of electromagnetic character, but it operates outside the filed consideration volume. The part of produced energy is dissipating into the heat $q(\vec{r},t)$. The pointed part is not electromagnetic and is not in the balance. In general case the warming-up of matter leads to nonequilibrium process, at that the heated up matter radiates in all spectrum, the process is not stationary and equilibrium and demands to solve the kinetic equation. Further we will consider this process as quasistationary and occurs under the constant temperature, i.e. we will consider the intensities of fields sufficiently small and the matter thermal capacity sufficiently great (infinitely large).

The momentum balance equation is conventionally received by each equations from (1) scalar multiplying on another field vector and by its summation using the identity $\vec{a}(\nabla \times \vec{b}) - \vec{b}(\nabla \times \vec{a}) = -\nabla \cdot (\vec{a} \times \vec{b})$. Here the point means the scalar product, and the symbol "×" means the crossproduct. Frequently we will omit the mentioned point (if there are not misunderstandings). As the result we have

$$\nabla \cdot \vec{S}(\vec{r},t) + \left[\vec{E}(\vec{r},t)\partial_t \vec{D}(\vec{r},t) + \vec{H}(\vec{r},t)\partial_t \vec{B}(\vec{r},t)\right] = -\left[\vec{E}(\vec{r},t)\vec{J}^e(\vec{r},t) + \vec{H}(\vec{r},t)\vec{J}^m(\vec{r},t)\right] . \tag{4}$$

The right-hand part of (4) is the power density which is spending by sources on field creation. This equation has the form $\nabla \cdot \vec{S}(\vec{r},t) + \partial_t w(\vec{r},t) = -\left[\vec{E}(\vec{r},t)\vec{J}^e(\vec{r},t) + \vec{H}(\vec{r},t)\vec{J}^m(\vec{r},t)\right]$, which is typical for balance [19].

The first term is the flow out power density of field, and the value $\partial_t w(\vec{r},t)$ is the accumulate power density of field and matter. In order to calculate the spend work the pointed value must be integrated:

$$w(\vec{r},t)=\partial_t^{-1}\left[\vec{E}(\vec{r},t')\partial_t\vec{D}(\vec{r},t')+\vec{H}(\vec{r},t')\partial_t\vec{B}(\vec{r},t')\right]. \quad (5)$$

Exactly this value, but not the term $\left[\vec{E}(\vec{r},t)\vec{D}(\vec{r},t)+\vec{H}(\vec{r},t)\vec{B}(\vec{r},t)\right]/2$ (as usually accepted) must be associated with $w(\vec{r},t)$ [16]. The energy (5) which spent on field and matter depends on all process prehistory, that is naturally for electrodynamics of continuum, whereas for vacuum electrodynamics it is not required [20]. Solving the equation (3) jointly with (1) one can determine all fields in the time interval $(0,t)$. It allows one to calculate the energy dissipation density $q(\vec{r},t)$ (the emitted heat of unite volume). Note that the dissipation is connected not only with conductivity, but with delayed polarization (i.e. with the delay of electrical displacement and magnetic induction responses on field influences). Finally for field-matter energy density we have $e(\vec{r},t)=w(\vec{r},t)-q(\vec{r},t)$. The energy transport velocity according to N. Umov concept [21] is the value $\vec{v}_e(\vec{r},t)=\vec{S}(\vec{r},t)/e(\vec{r},t)$. It is determined in each point for each time.

It is convenient to present the Maxwell equations (3) now in the form

$$\nabla\times\vec{H}(\vec{r},t)=\varepsilon_0\partial_t\vec{E}(\vec{r},t)+\vec{J}^e(\vec{r},t)+\vec{J}_P^e(\vec{r},t)\ ,$$

$$-\nabla\times\vec{E}(\vec{r},t)=\mu_0\partial_t\vec{H}(\vec{r},t)+\vec{J}^m(\vec{r},t)+\vec{J}_P^m(\vec{r},t),$$

and then the balance equations may be written as

$$\nabla\cdot\vec{S}(\vec{r},t)+\partial_t w_{(EM)}(\vec{r},t)=-\left[\vec{E}(\vec{r},t)\left(\vec{J}^e(\vec{r},t)+\vec{J}_P^e(\vec{r},t)\right)+\vec{H}(\vec{r},t)\left(\vec{J}^m(\vec{r},t)+\vec{J}_P^m(\vec{r},t)\right)\right], \quad (6)$$

$$w_{(EM)}(\vec{r},t)=\partial_t^{-1}\left(\varepsilon_0\vec{E}(\vec{r},t')\partial_t\vec{E}(\vec{r},t')+\mu_0\vec{H}(\vec{r},t')\partial_t\vec{H}(\vec{r},t')\right)=\left[\varepsilon_0\vec{E}^2(\vec{r},t)+\mu_0\vec{H}^2(\vec{r},t)\right]/2. \quad (7)$$

Since $\vec{S}$ is the field-matter power flow density, and the equation (6) is the same as for field excitation in vacuum by the sources as the sum of incident and polarization currents, that the sense of expression (7) is as field self-energy density. The quantity $w_{(EM)}$ in (6) is defined accurate within a constant. As the full field energy at $t=0$ is zero, the mentioned constant is zero only. The Pointing vector $\vec{S}$ is only defined within a arbitrary solenoidal vector $\vec{S}_0$, i.e. the vector which satisfy equation $\nabla\cdot\vec{S}_0(\vec{r},t)=0$ [19]. The flow of such vector over any closed surface is zero therefore its influence on common energy flow is absent. However it is easy to show that there is not any energy circulation over the closed paths (loops). As long as the field at $t=0$ was absent, the cited equation must be solved under the condition $\vec{S}_0(\vec{r},0)=0$, wherefrom one has $\vec{S}_0(\vec{r},t)=0$. In order to find the field self-energy transfer velocity, i.e. the energy connected only with photons (quasi-photons) [3], it is necessary to find the energy flow density of matter. If the problem of motion for matter particles in field is solved, then the average its velocity $\vec{v}(\vec{r},t)$ in physical infinitesimal volume may be determined. Then, if we choose any volume ΔV bounded by surface ΔS surrounding the point $\vec{r}$, we can defined in nonrelativistic limit the flow density in such a way:

$$\nabla\cdot\vec{S}_{(M)}(\vec{r},t)=\lim_{\Delta V\to 0}\frac{1}{2\Delta V}\oint_{\Delta S}\rho(\vec{r}',t)\vec{v}^2(\vec{r}',t)\vec{v}(\vec{r}')\cdot\vec{v}(\vec{r}',t)d^2r'.$$

Here the limit must be realized in terms of conversion to infinitesimal volume. This implies $\vec{S}_{(M)}(\vec{r},t)=\left[(1/2)\rho(\vec{r},t)\vec{v}^2(\vec{r},t)\right]\vec{v}(\vec{r},t)$, where $\vec{v}(\vec{r},t)$ is the velocity of matter and $\rho(\vec{r},t)$ is its density. Correspondingly we have $\vec{S}(\vec{r},t)=\vec{S}_{(M)}(\vec{r},t)+\vec{S}_{(EM)}(\vec{r},t)$. In relativistic case there are well-known relation between the energy and momentum [22,23] which must use. But this way is not very constructive as it demands to solve self-congruent dynamic and

excitation equations. On the microscopic level it is necessary to solve the excitation by incident sources and by polarization currents and also the equations for electrical and magnetic polarization vectors. Apparently, using only the balance equations, it is impossible to disjoint the energy or momentum balances on constituent parts for field and matter. For examination of this, let transform the equation (6) to the forms

$$\nabla \cdot \vec{S}_{(EM)}(\vec{r},t) + \partial_t w_{(EM)}(\vec{r},t) = -\left[\vec{E}(\vec{r},t)\vec{J}^e(\vec{r},t) + \vec{H}(\vec{r},t)\vec{J}^m(\vec{r},t)\right], \quad (8)$$

$$\nabla \cdot \vec{S}_{(EM)}(\vec{r},t) = \nabla \cdot \vec{S}(\vec{r},t) + \vec{E}(\vec{r},t)\vec{J}_P^e(\vec{r},t) + \vec{H}(\vec{r},t)\vec{J}_P^m(\vec{r},t). \quad (9)$$

The equation (8) is the same as in the vacuum. The question is thus: could the vector $\vec{S}_{(EM)}$ be unambiguously defined from equation (9)? According to Helmholtz theorem the vector field $\vec{S}_{(M)}$ is representable as sum of its potential and solenoidal parts:

$$\vec{S}_{(M)}(\vec{r},t) = \vec{S}(\vec{r},t) - \vec{S}_{(EM)}(\vec{r},t) = \nabla\Phi(\vec{r},t) + \nabla \times \vec{C}(\vec{r},t). \quad (10)$$

From here one has the Poisson equation for the value (10) determination:

$$\nabla^2\Phi(\vec{r},t) + \vec{E}(\vec{r},t)\vec{J}_P^e(\vec{r},t) + \vec{H}(\vec{r},t)\vec{J}_P^m(\vec{r},t) = 0 \ . \quad (11)$$

Notice, that this value of power flow density is defined accurate within the curl (rotor) of vector $\vec{C}(\vec{r},t)$. Also this rotor is the solenoidal vector and does not create the flow thus its flow over any closed surface is zero, however in general case $\vec{C}(\vec{r},t) \neq 0$. Moreover, it may be that $\vec{C}(\vec{r},0) \neq 0$, i.e. the flows in the matter may circulate at the time moment of field creation. Consequently, the vector (10) is ambiguously determined. In order to solve the equation (11) it is necessary to assume that the matter is located in some bounded volume *V* (that is the natural demand for stationary case). In this case if the $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ are known, the solution may be obtained, for example, by Green's function method $\Gamma(\vec{r},\vec{r}') = \left(4\pi|\vec{r}-\vec{r}'|\right)^{-1}$ for Poisson equation. Then the field self-energy is transferring with the velocity $\vec{v}_e^{(EM)}(\vec{r},t) = \vec{S}_{(EM)}(\vec{r},t)/w_{(EM)}(\vec{r},t)$. Evidently, this approach couldn't be spread on infinite medium. Furthermore, the velocity $\vec{v}_e^{(EM)}(\vec{r},t)$, as distinct from the velocity $\vec{v}_e(\vec{r},t)$, has not certain physical meaning. Let consider, for example, the plan monochromatic wave in the infinite medium. The amplitudes E_0 of electrical and H_0 of magnetic fields here are connected by the relation $H_0 = Z_0\rho_0 E_0$, where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$, and ρ_0 is the real normalized impedance. In general loss case the fields in such wave are shifted on phase angle φ. The conditions $\left\langle \vec{E}(\vec{r},t)\vec{J}_P^e(\vec{r},t)\right\rangle \approx 0$ and $\left\langle \vec{H}(\vec{r},t)\vec{J}_P^m(\vec{r},t)\right\rangle \approx 0$ may be fulfilled for monochromatic wave. Here the Dirac brackets mean the averaging over the period, i.e. the field on average does not exchange the energy with the matter. These equations are fulfilled religiously in the dispersionless medium, i.e. for ideal material conditions $\vec{D} = \varepsilon_0\varepsilon\vec{E}$, $\vec{B} = \mu_0\mu\vec{H}$. Then one may neglect the value (10) on average and calculate the terms $\left\langle \vec{S}\right\rangle$ and $\left\langle w_{(EM)}\right\rangle$. Let the plane wave propagates along *z*-axis with unite vector $\vec{z}_0$. Then we would obtain the expression

$$\vec{v}_e^{(EM)} \approx \vec{z}_0 \frac{2c\cos(\varphi)}{\rho_0 + 1/\rho_0} \leq c \quad (12)$$

for the transfer velocity of pure electromagnetic energy. In the case of considered material conditions we have: $\varphi = 0$, $\rho_0 = \sqrt{\mu/\varepsilon}$, $\vec{J}_P^e = \varepsilon_0(\varepsilon - 1)\partial_t\vec{E}$, $\vec{J}_P^m = \mu_0(\mu - 1)\partial_t\vec{H}$. In the case when $\rho_0 = 1$, i.e. at $\varepsilon = \mu$, the equation (12) gives the velocity of light, whereas $v_e = c/n = c/\varepsilon$. If one considers the balance relation (8), then it is clear that the second

term in the right part of (9) is normally adding to $w_{(EM)}$ and the full energy density $w = u = nw_{(EM)}$ is obtained, whereas $\vec{S}_{(EM)} = \vec{S}$, $\vec{S}_{(M)} = 0$.

Now let consider the momentum balance. We multiply vectorially the first equation in (3) at the left on the value $\vec{B}(\vec{r},t)$ and the second one on the value $\vec{D}(\vec{r},t)$ and subtract one from another:

$$\begin{aligned}\left[\vec{B}(\vec{r},t)\times\nabla\times\vec{H}(\vec{r},t)+\vec{D}(\vec{r},t)\times\nabla\times\vec{E}(\vec{r},t)\right]+\partial_t\left(\vec{D}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)=\\ =-\left[\vec{J}^e(\vec{r},t)\times\vec{B}(\vec{r},t)+\vec{D}(\vec{r},t)\times\vec{J}^m(\vec{r},t)\right]=-\vec{f}^L(\vec{r},t).\end{aligned} \tag{13}$$

The Lorentz force with inverse sign operating on incident currents stays in the right part of (13). I.e. this is the force spending by incident sources for full filed and matter momentum. The second term at the left in (13) is the time derivative of field-matter momentum density. Accordingly the same such density accurate within constant vector $\vec{g}_0^M(\vec{r})$ is $\vec{g}^M(\vec{r},t)=\vec{D}(\vec{r},t)\times\vec{B}(\vec{r},t)$, i.e. it must be taken in Minkowski form. The first term in (13) we rewrite in form

$$\nabla\cdot\hat{\Sigma}(\vec{r},t)=\partial_\nu\hat{\Sigma}^\nu_{\nu'}(\vec{r},t)=\left[\vec{B}(\vec{r},t)\times\nabla\times\vec{H}(\vec{r},t)+\vec{D}(\vec{r},t)\times\nabla\times\vec{E}(\vec{r},t)\right]. \tag{14}$$

Here $\nu = x,y,z$. The value $\hat{\Sigma}(\vec{r},t)$ is the second rank tensor in the tridimensional space. Therefore its divergence (or the furl over one index) is the vector staying in the right part of (14). The indicated tensor is also defined accurate within arbitrary tensor which satisfies equation $\nabla\cdot\hat{\Sigma}_0(\vec{r},t)=0$. Because the field was absent at the time $t=t_0=0$, it is necessary to impose the initial conditions so $\vec{g}_0^M(\vec{r})=0$, $\hat{\Sigma}_0(\vec{r},0)=0$. The values $\vec{g}^M(\vec{r},t)$ and $\hat{\Sigma}(\vec{r},t)$ under this conditions are unambiguously determined by way of solutions of excitation problem in any time, i.e. through the fields $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$. At that in order to determine the $\vec{g}^M(\vec{r},t)$ one must calculate the integrals (1), and for the determination of $\hat{\Sigma}(\vec{r},t)$ it is necessary to solve the differential equation (14) yet. For this solution one can use the Helmholtz theorem and solve the Poisson equation. Thus, the momentum balance equation is:

$$\partial_\nu\hat{\Sigma}^\nu_{\nu'}(\vec{r},t)+\partial_t g^M_{\nu'}=-f^L_{\nu'}=-\partial_t\partial_t^{-1}f^L_{\nu'},\quad \nu'=x,y,z. \tag{15}$$

Here the first term is the full momentum component ν' flow. It is follows from this equation that the transfer velocity of filed-matter momentum ν' - component is [19]

$$v_{i\nu'}=\partial_\nu\hat{\Sigma}^\nu_{\nu'}(\vec{r},t)/\,g^M_{\nu'}, \tag{16}$$

and $g^M_{\nu'}$ is the full created momentum. If we consider the full infinite space and the volume confined by the sphere surface with the radius $r=ct$ there, then the full momentum of field, matter and source is conserved: $G_{\nu'}=\partial_{\vec{r}}^{-1}g_{\nu'}=\partial_{\vec{r}}^{-1}\left(g^M_{\nu'}+\partial_t^{-1}f^L_{\nu'}\right)=0$, as the flow over the sphere is zero. Let consider the transmitted to matter momentum. It is obvious that such transmission is carrying out by polarization currents: $\vec{J}^e_P=\partial_t\left(\vec{D}-\varepsilon_0\vec{E}\right)=\sigma\vec{E}+N^e_p\partial_t\vec{p}^e$ and $\vec{J}^m_P=\partial_t\left(\vec{B}-\mu_0\vec{H}\right)=N^m_p\partial_t\vec{p}^m$. As stated above yet, the conductivity current $\sigma\vec{E}=e\vec{v}N$ is taken into account. Here the N is the number of charges in unit volume. Correspondingly N^e_p and N^e_p are the numbers of electric and magnetic dipoles with dipole moments $\vec{p}^e$ and $\vec{p}^m$. The specific momentum transferred to the matter is

$$\vec{G}_{(M)}=\partial_t^{-1}\vec{g}_{(M)},\qquad \vec{g}_{(M)}(\vec{r},t)=\vec{J}^e_P(\vec{r},t)\times\vec{B}(\vec{r},t)+\vec{D}(\vec{r},t)\times\vec{J}^m_P(\vec{r},t). \tag{17}$$

It remains to find the matter flow density. For this we rewrite the equations (3) as

$$\nabla\times\vec{H}(\vec{r},t)=\varepsilon_0\partial_t\vec{E}(\vec{r},t)+\vec{J}_P^{\,e}(\vec{r},t)+\vec{J}^{\,e}(\vec{r},t)\ ,\ -\nabla\times\vec{E}(\vec{r},t)=\mu_0\partial_t\vec{H}(\vec{r},t)+\vec{J}_P^{\,m}(\vec{r},t)+\vec{J}^{\,m}(\vec{r},t) \tag{18}$$

and present the momentum balance for this Maxwell equations form. Multiplying the first equation in (18) at the left on the value $\mu_0\vec{H}(\vec{r},t)$, the second one on the value $\varepsilon_0\vec{E}(\vec{r},t)$ and subtracting one from another we have:

$$\nabla\cdot\hat{\Sigma}_{(EM)}(\vec{r},t)+\partial_t\vec{g}^{\,A}(\vec{r},t)=-\vec{f}^{\,L}(\vec{r},t)\ . \tag{19}$$

Here the Lorentz force

$$\vec{f}^{\,L}_{(M)}(\vec{r},t)=\vec{J}_P^{\,e}(\vec{r},t)\times\vec{B}(\vec{r},t)+\vec{D}(\vec{r},t)\times\vec{J}_P^{\,m}(\vec{r},t)$$

acted on the matter is transferred into the left part of (19) and is presented by any flow density $\hat{\Sigma}_{(EM)}$. This density satisfies the differential equation:

$$\nabla\cdot\hat{\Sigma}_{(EM)}(\vec{r},t)=\left[\mu_0\vec{H}(\vec{r},t)\times\nabla\times\vec{H}(\vec{r},t)+\varepsilon_0\vec{E}(\vec{r},t)\times\nabla\times\vec{E}(\vec{r},t)\right]+\vec{f}^{\,L}_{(M)}(\vec{r},t)=$$
$$=\mu_0\left[\nabla\vec{H}^2(\vec{r},t)/2-\left(\vec{H}(\vec{r},t)\cdot\nabla\right)\vec{H}(\vec{r},t)\right]+\varepsilon_0\left[\nabla\vec{E}^2(\vec{r},t)/2-\left(\vec{E}(\vec{r},t)\cdot\nabla\right)\vec{E}(\vec{r},t)\right]+\vec{f}^{\,L}_{(M)}(\vec{r},t). \tag{20}$$

Let consider the means of balance (19). The Abraham density $\vec{g}^{\,A}(\vec{r},t)$ is the self electromagnetic filed momentum density. It is creating by primary or initial sources and by secondary sources (or polarization currents) which define the density $\hat{\Sigma}_{(EM)}(\vec{r},t)$. When the sources are absent ($\vec{f}^{\,L}=0$), then the equation (19) is typical conservation law. So, the tensor value $\hat{\Sigma}_{(EM)}(\vec{r},t)$ determines the self field momentum density flow. It is presented accurate within some divergentless or solenoidal tensor $\hat{\Sigma}^{(0)}_{(EM)}(\vec{r},t)$ and under the initial condition $\hat{\Sigma}^{(0)}_{(EM)}(\vec{r},0)=0$. The matter momentum flow is determined by the tensor $\hat{\Sigma}_{(M)}(\vec{r},t)=\hat{\Sigma}(\vec{r},t)-\hat{\Sigma}_{(EM)}(\vec{r},t)$. Now we can obtain the transfer velocity filed and matter self momentum correspondingly:

$$v^{(em)}_{i\nu'}=\partial_\nu\hat{\Sigma}_{(EM)}{}^{\nu}_{\nu'}(\vec{r},t)/\,g^{A}_{\nu'}(\vec{r},t)\ ,\qquad v^{(M)}_{i\nu'}=\partial_\nu\hat{\Sigma}_{(M)}{}^{\nu}_{\nu'}(\vec{r},t)/\,g_{(M)\nu'}(\vec{r},t)\ . \tag{21}$$

In order to transform the equation (20) we have used the vector identity $\nabla\left(\vec{a}\cdot\vec{b}\right)=\left(\vec{a}\cdot\nabla\right)\vec{b}+\left(\vec{b}\cdot\nabla\right)\vec{a}+\vec{a}\times\nabla\times\vec{b}+\vec{b}\times\nabla\times\vec{a}$, which at $\vec{a}=\vec{b}$ takes the form $\nabla\vec{a}^2=2\left(\vec{a}\cdot\nabla\right)\vec{a}+2\vec{a}\times\nabla\times\vec{a}$. Also in order to transform the introduced tensors, for example, the tensor $\hat{\Sigma}(\vec{r},t)$, one may use the vector-tensor identity

$$\vec{a}\times\left(\nabla\times\vec{b}\right)+\vec{b}\times\left(\nabla\times\vec{a}\right)=\nabla\cdot\left[\hat{I}\left(\vec{a}\vec{b}\right)-\vec{a}\otimes\vec{b}-\vec{b}\otimes\vec{a}\right]+\vec{a}\left(\nabla\cdot\vec{b}\right)+\vec{b}\left(\nabla\cdot\vec{a}\right).$$

For the equal vectors it takes the form

$$2\vec{a}\times\left(\nabla\times\vec{a}\right)=\nabla\cdot\left[\hat{I}\vec{a}^2-2\vec{a}\otimes\vec{a}\right]+2\vec{a}\left(\nabla\cdot\vec{a}\right).$$

The tensor $\hat{\Sigma}(\vec{r},t)$ for vacuum is equal to Maxwell stress tensor $\hat{\sigma}^{\nu}_{\nu'}$ which is taken with inverse sign. It also may be transformed taking into consideration with regarding that according to (3)

$$\nabla\cdot\vec{D}(\vec{r},t)=-\partial_t^{-1}\left(\nabla\cdot\vec{J}^{\,e}(\vec{r},t')\right)=\partial_t^{-1}\left(\partial_{t'}\rho^e(\vec{r},t')\right)=\rho^e(\vec{r},t),$$
$$\nabla\cdot\vec{B}(\vec{r},t)=-\partial_t^{-1}\left(\nabla\cdot\vec{J}^{\,m}(\vec{r},t')\right)=\partial_t^{-1}\left(\partial_{t'}\rho^m(\vec{r},t')\right)=\rho^m(\vec{r},t).$$

Since the incident sources satisfy the continuity equation (charge conservation law): $\nabla\cdot\vec{J}^{\,e}(\vec{r},t)+\partial_t\rho^e(\vec{r},t)=0$, $\nabla\cdot\vec{J}^{\,m}(\vec{r},t)+\partial_t\rho^m(\vec{r},t)=0$. As the incident magnetic charges do not exist, i.e. $\rho^m(\vec{r},t)=0$, that the density of incident magnetic current is solenoidal and presentable as the rotor of some electrical current density.

Thus, if the momentum density is defined according to Minkowski, it in nondispersive medium is the density of the field-matter substance, and its transfer velocity is the phase velocity. Its generalization for the dispersive media leads to transfer velocity (16) which is the full momentum transfer velocity of field-matter. In this case it is necessary for the determination of all values to solve the nonstationary excitation problem, and all considered above magnitudes are depending on the process prehistory, i.e. they may be complicated dependent on time. It must be note that the obtained local (differential) balance relations may be rewritten as integral relations for any volume V. Then the integrals form u and $\vec{g}^M$ over the mentioned volume present correspondingly the full energy U and the full momentum $\vec{G}^M$ of this volume. These values are conserved from the viewpoint of global conservation law. There may be two cases. 1) The sources of field are located in the volume. Then the full energy and momentum balances present itself the inhomogeneous balance relations with values in its right parts, which corresponding to production of energy and momentum in the volume. The negative energy production means the dissipation. 2) There were no any sources in the volume before the considered time t. In this case the sources are located outside the volume, and one can take as the time t_0 the instant of time when the field comes to the volume. In this case the energy and momentum are conserved from such viewpoint, that the value $\partial_t(U+Q)$ in each time is equal to the power flowing out the volume, and the change of full momentum is equal to flowing momentum. For the stationary (monochromatic) in time field or wave for some dispersion laws one may obtain concrete kinds of mentioned values. In this case, using the limit transfer from quasistationary process to stationary one, the averaged over the period densities "forget its initial senses", i.e. do not depend on them. All received values for EMT, energy and momentum densities coincide with the Abraham forms. As it is easy to see, all received values are defined unambiguously. So, the solution of differential equations $\nabla\cdot\hat{\Sigma}_0(\vec{r},t)=0$ with zero initial conditions gives zero of tensor $\hat{\Sigma}_0$ components.

3. Plane monochromatic wave in the conduction magnetodieleectric medium

For more concrete and simple analysis we will consider the monochromatic plane wave falls down on the magnetodielectric layer with thickness d. Let the layer is located at the region $0\le z\le d$ and has the constant real spectral permittivity ε' and permeability μ, at that time $\varepsilon',\mu\ge 1$. This means the absence of frequency dispersion, that is justly in any frequency region $0<\omega<<\omega_{\min}$, where $\omega_{\min}$ is any minimal frequency from the set of self resonant mater frequencies, the frequency of normal skin-effect violation (if such takes place), and also the plasma frequency ω_p of uncombined carriers of charge. As we consider such, we suppose the conducting medium, i.e. having the complex spectral permittivity with the pole at zero frequency [17]:

$$\varepsilon(\omega)=\varepsilon'-j\varepsilon''=\varepsilon'-j\sigma/(\varepsilon_0\omega)\,, \qquad (22)$$

and constant permeability $\mu=const$. For example, for the water we have such frequency $\omega_{\min}\sim\omega_c=1/\tau$ which lays at 100 GHz, ω_c is the collision frequency, and τ determines the relaxation time in the Debye's formula. For the metals the frequency $\omega_{\min}$ may lay in the diapason from infra-red to ultra-violet [24,25]. Further we will consider the plane quasi-monochromatic wave (long wave train or wave packet), which is approached from the left at time $t_0=0$ to the layer boundary and is excited at infinity at the time $t=-\infty$. Such plane wave may be excited in both sides formally mathematically by the current sheet with the density $\vec{J}^e_{inc}(x,y,z,t)=\vec{x}_0\chi(t+\tau)I_x(t)\delta(z+l)$, which is located at $z=-l$ and is started to act

at the time $t=-\tau$ [26]. Here χ is the Heaviside step function, and for monochromatic wave it is convenient to take $I_x(t)=\sin(\omega t)$. Later we will consider the wave train diffraction on the plate (layer). But at first let consider the plane quasi-monochromatic wave. For it we introduce the current density in such a way:

$$\vec{J}^{e}_{inc}(x,y,z,t)=\vec{x}_0\chi(t+\tau)I_x\delta(z+l)[1-\exp(-\delta t)]\sin(\omega t)\ , \qquad (23)$$

and consider the field at great time $t>>\tau+l/c$. As the special case we also will consider the excitation by the current density (23) in the infinite homogeneous conducting magnetodielectric medium. The introduced current densities create the plane wave with the components $E_x=E$ and $H_y=H$ (further we often will omit the indexes x and y). The indicated wave satisfy the Maxwell equations in the form

$$\partial_z H=-\varepsilon_0\varepsilon'\partial_t E-\sigma E\ , \qquad \partial_z E=-\mu_0\mu\partial_t H\ . \qquad (24)$$

For vacuum we impose $\varepsilon'=\mu=1$, $\sigma=0$.

As is well known, two velocities, which defined the transfer of inherent wave physical substances or characteristic, may be introduced for plane monochromatic wave with complex dependence $\exp(j(\omega t-\beta(\omega)z)-\alpha(\omega)z)$ moving along z-axis in dispersive medium. These are the energy transport velocity v_e and the momentum transport velocity v_i [27–29] (owing to one-dimensionality we omit the vector designations). Besides, for the dispersion law $\beta(\omega)$ one may else introduce two velocities which determine the movement of mathematical (kinematic) wave characteristics. These are the phase velocity $v_p(\omega)=\omega/\beta(\omega)$ and the group velocity $v_g(\omega)=(\partial\beta(\omega)/\partial\omega)^{-1}$. The first one characterizes the phase movement, and the second one defines the velocity of phase perturbation or interference pattern (beating) for two infinitely frequency closed waves with equal amplitudes (as it was introduces by Stokes). At the same time the positive derivative corresponds to positive dispersion or forward wave (in z-direction), and negative one – to negative dispersion or backward (inverse) wave. The coefficient $n'(\omega)=c/v_p(\omega)$ determines the wave retardation (deceleration) and the refraction on the boundary interface, and the coefficient $n''(\omega)$ corresponds with loss. The normal dispersion corresponds to relation $\partial n'(\omega)/\partial\omega>0$ and $\partial v_p(\omega)/\partial\omega<0$, whereas the anomalous one occurs when $\partial n'(\omega)/\partial\omega<0$ and $\partial v_p(\omega)/\partial\omega>0$ [30]. Two last velocities are not in keeping with movement of any physical substances and are only opportune mathematical conception attached to wave descriptions [14–16,31–41], although frequently $v_g(\omega)$ is frequently identified with $v_e(\omega)$, that is incorrectly for dissipative media [14–16,33–35,39–40]. As all real media are dissipative in some way, the relation $v_g(\omega)=v_e(\omega)$ is fulfilled only for several ideal models, for example, in the ideal collisionless plasma. The dispersion in ideally conducting waveguides, including the periodic waveguides or slow-wave systems, presents the quite another matter: for harmonic wave there are not the frequency spectral wave train, and the dispersion law arises owing to spectral series of partial waves, which are moving on-the-mitre θ to the waveguide axis with the phase velocity equal to velocity of light. The mentioned wave series in general case depends on two angles, one of which defines the energy transfer velocity along the axis, and another may has the continuous spectrum of meanings [16]. Under this we have the geometric relations $v_p=c/\cos(\theta)$, $v_g=c\cos(\theta)$, and $v_p v_g=c^2$. This case is trivial; it corresponds to electromagnetic structures, but not to media, and here is not considered. In a number of works, for example, in [41] it is asserted that always fulfills $v_g=v_e\le c$, though this is not true [14–16,39,40].

But ibidem for waves with negative energy may fulfill $v_g \to \infty$ [41]. Strictly speaking, one can introduce infinite number of values having the dimension of a quantity as velocity in the problem of electromagnetic wave train propagation. For example, we can consider following values: $v_n = \left(\omega^{n-1}\partial^n \beta(\omega)/\partial\omega^n\right)^{-1}$. For mathematical description the appropriate complex velocities often have been used. In this case the value β is replaced by complex propagation constant $\gamma(\omega) = \beta(\omega) - j\alpha(\omega)$ [15,37] and correspondingly the complex refractive index $n(\omega) = n'(\omega) - jn''(\omega)$ is introduced [17]. The complex phase velocity of signal may has the mathematical sense, as against to complex v_g, which may be introduces only for complex signal. The group velocity 4-vector proposed in the paper [42], which has the meaning only for self-adjoint Hamiltonian when the Lagrangian is the quadratic function of generalized coordinates and impulses. In literature there is well-known the Leontovich-Lighthill theorem for conservative (nondissipative) systems [27–29, 31, 43–45], according to which under the indicated Hamiltonian the relation $v_g = v_e$ takes place. In our case (and in general for dissipative media) the frameworks of this theorem are newer implemented.

Traditionally the velocity v_g is introduced by way of phase constant $\beta(\omega)$ decomposition in Taylor's series about any frequency (for example, carrying one) in the spectral integral with the abandonment of zeroth-order and и first-order terms (the first approach of dispersion theory) [36–38]. Sometimes the inverse decomposition $\omega(\beta)$ is used [35]. The taking into account of high-order terms just leads to arising of mentioned velocities. The second derivative $\partial^2\beta/\partial\omega^2$ in that case is as a first approximation which characterizes the velocity of wave train smearing as whole [33–37]. In dissipative media the analogous complex velocities may be considered when they are also the functions of attenuation constant $\alpha(\omega)$. These decompositions are asymptotic [38,46], i.e. not obligatory convergent. From the beginning we will consider not wave train propagation, but the simple one-dimensional plane monochromatic wave case. In such wave there is no frequency wave group, and so no reason to introduce the group velocity (although formally for dispersion $\beta(\omega)$ it may be defined, that we do).

Often it is necessary to consider the conducting media at sufficiently low frequencies, when the non-connected with conductivity dispersion may be neglected. For example, these are the semi-conductors and metals at radio and microwave frequencies, the see water in radio-frequency band, ionospheric plasma (such as Heaviside layer) under the ultra low frequencies. The mentioned law, when the dispersion is determined only by frequency nondependent conductivity σ (i.e. by direct current conductivity), has the form [35]

$$\beta(\omega) = (\omega/c)\sqrt{\varepsilon'\mu/2\left[1+\sqrt{1+\sigma^2/\left(\varepsilon_0^2\varepsilon'^2\omega^2\right)}\right]} = \omega n'(\omega)/c \ , \tag{25}$$

$$\alpha(\omega) = (\omega/c)\sqrt{\varepsilon'\mu/2\left[-1+\sqrt{1+\sigma^2/\left(\varepsilon_0^2\varepsilon'^2\omega^2\right)}\right]} = \omega n''(\omega)/c \ . \tag{26}$$

Already the direct application of formula (22) demonstrates that the group velocity may exceed the value $c/\tilde{n}$, where the $\tilde{n} = \sqrt{\varepsilon'\mu}$ is the retardation coefficient or refractive index in the medium without conductivity. Moreover, it may exceed the velocity of light in vacuum *c*. Let us assume that $\mu = 1$. Signifying $\tilde{\omega} = \sigma/(\varepsilon_0\varepsilon)$, one has

$$\frac{d}{d\omega}\beta(\omega) = \frac{\beta(\omega)}{\omega}\left\{1 - \frac{(\tilde{\omega}/\omega)^2}{2\left(1+\sqrt{1+(\tilde{\omega}/\omega)^2}\right)\sqrt{1+(\tilde{\omega}/\omega)^2}}\right\}_{\omega=\tilde{\omega}} = \frac{0.8535...}{v_p(\tilde{\omega})} = \frac{\tilde{n}/c}{1.0663..} \ . \tag{27}$$

The mentioned excess comes at $\widetilde{n} < 1.0663...$, that may be, for example, in weakly ionized air under the conditions $\omega_c >> \omega_p$ and $\omega_c >> \widetilde{\omega}$. The consideration of permittivity dispersion in a number of cases allows one to make this excess still more essential [16]. We refer to two works [39,40] which among the first register the fact that group velocity may exceed the velocity of light in the region with anomalous dispersion (see also [17]), and at the same time show that the signal at that time propagates with velocity $v < c$. Consequently, in conducting medium the value v_g certainly not characterizes the energy transportation velocity. On the contrary, the phase velocity in such medium with anomalous positive dispersion always letter than velocity of light: $v_p(\omega) = c / n'(\omega) < c$.

The main goal of this paper is the proof the relations $v_e = v_p$ and $v_i = v_p$ for considerable case. Let observe that in the paper [16] the relation $v_e = v_p$ has been proved for the media with anomalous positive dispersion which permittivity is described by Debye's formula (i.e. for the polar dielectrics with hard dipoles). This has been proved using two independent methods. We will regard that the process of wave propagation is quasi-equilibrium, i.e. existent without a heating and under a constant temperature.

4. Electromagnetic energy density in monochromatic wave

Let the plane linearly polarized monochromatic wave with the electric filed component E_x propagates in conducting medium. The indicated wave creates the conductivity current density $J_{x\sigma} = \sigma E_x$, which leads to wave energy dissipation. This dissipation has the exponential distribution along *z*-axis in form $\exp(-2\alpha(\omega)z)$ and creates inhomogeneous along *z* heating of infinite space. Such heating, for one's turn, generates the thermal radiation in both directions $\pm z$ with all spectral components. Hence, the process primordially is nonequilibrium. It may be considered approximately as equilibrium and single-frequency, if the amplitude of wave is small (or infinitely small), or if the thermal capacity of medium is infinitely large.

The undamped harmonic wave in infinite dissipative media may propagate only due to the distributed incident source energy which compensates the loss on dissipative heat *Q* [17]. We propose that such sources are outside the zone of wave consideration (usually at the infinity). The energy of plane wave is infinite even for lossless case that characterizes this wave as convenient mathematical abstraction (the solution of homogeneous Maxwell equations). Formally mathematically the plane wave is excited in both directions by the electrical current sheet with the density (23), which operates infinitely long in time. If the finite sources are at the infinity then the plane wave is the limit case of spherical wave.

The permittivity (22) is obtained by exponential filed dependencies substitution direct to Maxwell equations with taking into account of conductivity current. One may get another derivation of this value. Namely, it is necessary to count the average over a period polarization of unite volume and use the relation

$$D_x(\omega,t,z) = D(\omega,t,z) = \varepsilon_0\varepsilon(\omega)E(\omega,t,z) = \varepsilon_0 E(\omega,t,z) + P_x(\omega,t,z) \ . \qquad (28)$$

Here $E(\omega,t,z) = E_x(\omega,0,0)\exp(j\omega t - j\gamma(\omega)z)$ is denoted. Let symbolize $E_0 = E_x(\omega,0,0)$. The relation (28) must be averaged over the oscillation period. In our case it is $\langle P_x(\omega,t,z)\rangle = (\kappa_1 + \kappa_2(\omega))\langle E(\omega,t,z)\rangle$. It may be written for the introduced receptivities $\kappa_1 = \varepsilon' - 1$ and $\kappa_2(\omega) = -j\sigma/(\varepsilon_0\omega)$. Indeed, the unite volume polarization in this medium is creating by proper matter polarization and by the movement of free charges, which are scattering on matter atoms, molecules, and one on another. The first polarization happens instantly without any delay owing to our assumption that the characteristic matter frequencies are extremely high. The charge motion is described by the equation

$\dot{x}(t) = \sigma(Ne)^{-1} E_x(\omega, t, z)$ and occurs in such a way that their potential energy is equal to zero, and the average kinetic energy has the form $\langle U_K \rangle = m\sigma^2 E_0^2 / (4Ne^2) \exp(-\alpha(\omega) z)$, where $\sigma = Ne^2 / (m\omega_c)$, N is the number of charged particles of unite volume. This result also may be gotten from the plasma dielectric receptivity

$$\kappa_p(\omega) = -\frac{\omega_p^2}{\omega(\omega - j\omega_c)} \tag{29}$$

in the supposition that $\omega << \omega_p$ and $\omega << \omega_c$, i.e. that the plasmic frequency and the collision frequency are highly large, and at the same time $\sigma = \varepsilon_0 \omega_p^2 / \omega_c$ is the conductivity on zero frequency. In other words, the conductive medium may be viewed as the plasma at very low frequencies. The averaged electrical part of energy density and the permittivity for gas of oscillators with the self-resonant frequency ω_0 have been obtained in the work [15] and have the forms

$$\langle U_E(t,z) \rangle = \frac{\varepsilon_0 E_0^2}{4} \left\{ 1 + \frac{\omega_p^2 (\omega^2 + \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + \omega_c^2 \omega^2} \right\} \exp(-2\alpha(\omega) z) \ . \tag{30}$$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2 (\omega^2 - \omega_0^2 + j\omega\omega_c)}{(\omega^2 - \omega_0^2)^2 + \omega_c^2 \omega^2} \ . \tag{31}$$

It must put $\omega << \omega_0$ (or $\omega = 0$) for nonconductive medium without the dissipation, that gives

$$\varepsilon(\omega) = 1 + \omega_{p1}^2 / \omega_0^2 = 1 + \kappa_1 = \varepsilon' = const \ . \tag{32}$$

$$\langle U_E(t), z \rangle = (1/4) \varepsilon_0 E_0^2 (1 + \omega_{p1}^2 / \omega_0^2) \exp(-2\alpha(\omega) z) = (1/4) \varepsilon_0 \varepsilon E_0^2 \exp(-2\alpha(\omega) z) \ . \tag{33}$$

Here the index 1 denotes the plasmic frequency connected with the matter dipole concentration. In such medium the energy at the frequencies lower than $\omega_{\min}$ propagates with phase velocity. For conducting medium with plasma it is necessary to add the terms from the formulas (30), (31), in which $\omega_0 = 0$ (free charges). In this case we have $\omega_{p2} >> \omega$, therefore

$$\langle U_E(t,z) \rangle = \frac{\varepsilon_0 E_0^2}{4} (1 + \omega_{p1}^2 / \omega_0^2 + \omega_{p2}^2 / \omega_c^2) \exp(-2\alpha z) \ , \tag{34}$$

$$\varepsilon(\omega) = 1 + \omega_{p1}^2 / \omega_0^2 - j\omega_{p2}^2 / (\omega_c \omega) = 1 + \kappa_1 + \kappa_2 \ . \tag{35}$$

5. Velocity of energy movement

For any wave process the transfer velocity of some substance is determined according to [21] by its density and by vector of flow density in unit time. In our case for the energy it is the Pointing vector $\vec{S}(z,t) = \vec{z}_0 S(z,t)$, and at that:

$$\vec{v}_e(z,t) = \vec{z}_0 v_e(z,t) = \vec{S}(z,t) / u(z,t) = \vec{z}_0 E(z,t) H(z,t) / u(z,t). \tag{36}$$

This leads from the energy conservation law (4) under the general suppositions. The real physical fields are used in (36) and further. For harmonic fields in homogeneous medium averaging the (36) over the time with consideration of (30) and also taking the magnetic part of energy, we get

$$v_e(\omega) = c \frac{\sqrt{2}}{\sqrt{\varepsilon' \mu}} \frac{\sqrt{1+\delta}}{1+\varsigma+\delta} < v_p(\omega). \tag{37}$$

Here $\delta = \sqrt{1+\varsigma^2}$, $\varsigma = \sigma/(\varepsilon_0\varepsilon'\omega)$. If the conditions $\omega << \omega_c$ and $\sigma >> \varepsilon_0\varepsilon'\omega_c$ are fulfilled, then one can neglect of second term in denominator of (37) and get

$$v_e(\omega) \approx c\frac{\sqrt{2}}{\sqrt{\varepsilon'\mu}\sqrt{1+\delta}} = v_p(\omega). \tag{38}$$

Under the indicated approximations we also may neglect of unit under the square root, and then

$$v_e(\omega) = v_p(\omega) = \frac{c}{\sqrt{\varepsilon'\mu}}\sqrt{\frac{2\varepsilon_0\varepsilon'\omega}{\sigma}} = \sqrt{\frac{2\omega}{\mu_0\mu\sigma}} = v_g(\omega)/2\,. \tag{39}$$

The group velocity may exceed the velocity of light in vacuum for the dispersion (39) and for $\mu = 1$ at small decelerations. Namely, from the demand $v_g > c$ we have the condition $\sigma < 8\omega\varepsilon_0$. Using the conditions of formula (39), we have $\omega\varepsilon_0\varepsilon << \sigma < 8\omega\varepsilon_0$. For the media with $\varepsilon \sim 1$ it may regard such when the conductivity σ is closed to $8\omega\varepsilon_0$ from below. But for the frequencies of order $\sigma/(8\varepsilon_0)$ one couldn't fully neglect the bias current as compared with conductivity current, and so should consider the rigorous relation (25).

There is another wave to obtain the result (38). This is limit transfer from quasi-stationary process to stationary one [16]. Let the plane surface current source has been arisen at $z = 0$ and the time $t = 0$ (23). The field was absent for times $t < 0$. The source creates the plane wave in the region $|z| > 0$. Then we can write its work (energy) density w spend on the field creation (5) in our case for the clearness as:

$$w(z,t) = \int_0^t \{E(z,t')\partial_{t'}D(z,t') + H(z,t')\partial_{t'}B(z,t')\}dt' =$$

$$= \frac{\varepsilon_0 E^2(z,t) + \mu_0 H^2(z,t)}{2} + \int_0^t \{\varepsilon_0\hat{\kappa}^e(0,z)E^2(z,t') + \mu_0\hat{\kappa}^m(0,\vec{r})H^2(z,t') + \tag{40}$$

$$+ \int_0^{t'} \left[\varepsilon_0 E(z,t')\partial_t\hat{\kappa}^e(t'-t'',z)\vec{E}(t'',z) + \mu_0\vec{H}(z,t')\partial_t\hat{\kappa}^m(t'-t'',z)\vec{H}(z,t'')\right]dt''\}dt'.$$

Here the kernels of permittivity and permeability integral operators $\hat{\kappa}^e$ и $\hat{\kappa}^m$ have been symbolized. Further on account of homogeneity the dependence on z is omitted, and the fields are exponential-dependent on z. Note, that low limit in (40) may be taken as $-\infty$. For the considered dispersion law we have $\hat{\kappa}^m(t) = (\mu - 1)\delta(t)$, and the operator of permittivity has the presentation

$$\hat{\kappa}^e(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\omega)\exp(j\omega t)d\omega = (\varepsilon' - 1)\delta(t) + \chi(t)\sigma/\varepsilon_0\,,$$

where $\chi(t)$ is the Heaviside step function. In order to find the energy density there is need to subtract the dissipative energy density or heat q from (40) [15–17]. Detecting these values at big times $t >> 1/\delta$ and $t >> 1/\omega$, one can get mentioned parameters for quasi-monochromatic process. Averaging over the oscillation period $2\pi/\omega$ and taking the limit $t \to \infty$, we can find the energy of such process. Substituting all incoming value expressions into integrals like (40) over the respective spectral integrals, extracting the proper delta-functions and integrating with them over the time, and then calculating the spectral integral by the method of residuals, we get the formula (30). Note, that the result $\langle U_E \rangle = \varepsilon_0\varepsilon'|\vec{E}|^2/4$ for the conducting medium ($\omega_0 = 0$) at the low frequencies just follows from Pointing theorem in complex form [34,35]. This result also is obtained from the explicit field representations by the equations (24) [35]:

$$E_x = E = E_0\cos(\omega t - \beta z)\exp(-\alpha z), \qquad H_y = H = H_0\cos(\omega t - \beta z - \varphi)\exp(-\alpha z)\ . \qquad (41)$$

Here $\varphi = \arctan(\alpha/\beta)$ is the phase angle (shift), which is given by the formula (6.32) from the work [35], and the ratio of amplitudes in (18) gives the real impedance $Z = E_0/H_0 = \omega\mu_0\mu/\sqrt{\alpha^2+\beta^2}$. At the same time $Z = Z_0\sqrt{\mu/\sqrt{\varepsilon'^2+\sigma^2/(\omega^2\varepsilon_0^2)}}$, $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ (formula (6.31) from [35]). This implies

$$v_e = \frac{\langle S\rangle}{\langle U\rangle} = 2\frac{E_0^2\cos(\varphi)/Z}{\varepsilon_0\varepsilon' E_0^2 + \mu_0\mu E_0^2/Z^2} = \frac{2}{\omega\mu_0\mu}\frac{\beta}{\varepsilon_0\varepsilon' + \varepsilon_0\varepsilon'\sqrt{1+\varsigma^2}} = \frac{2}{\omega^2}\frac{\beta\omega}{\varepsilon_0\varepsilon'\mu_0\mu(1+\delta)} = \frac{\omega}{\beta} = v_p\ .$$

6. Momentum transfer velocity

Let examine the question about the electromagnetic momentum transfer velocity. In vacuum the momentum density is unambiguously determined by the Abraham vector $\vec{g}^{\,A}(t,z) = \vec{z}_0 S(t,z)/c^2$. In the electromagnetic of continuity there are traditionally two forms: the Abraham $\vec{g}^{\,A}$ and the Minkowski $\vec{g}^{\,M}(t,z) = \vec{z}_0\tilde{n}^2 S(t,z)/c^2 = \tilde{n}^2\vec{g}^{\,A}(t,z)$ ones, at that up to now the question for benefit of choice one from them has not solved yet [3–13,41]. This fact even is reflected in the physical encyclopedia. Both definitions lead to identical conservation laws for Minkowski and Abraham energy-momentum tensors (EMT) in dispersionless media [3]. It is considered that the chose of form for momentum density impossible without the solution of equations of substance motion in the filed and before the definition of substance EMT. However, without the momentum density and its flow definitions and also even one from them it is impossible to define the momentum transfer velocity also. Such ambiguities are generally typical for electromagnetics of continuum. Thus, the introduction of electromagnetic potentials in media is also ambiguous or polysemantic [47] that is connected not only with calibration transformations. In our case of plane wave $\vec{g}^{\,M}(t,z)$ and $\vec{g}^{\,A}(t,z)$ have the forms

$$\vec{g}^{\,A}(t,z) = \vec{z}_0 g^{\,A}(t,z) = \frac{\vec{z}_0 E_0^2}{c^2 Z}\cos(\omega t - \beta(\omega)z)\cos(\omega t - \beta(\omega)z - \varphi)\exp(-2\alpha(\omega)z)\ , \qquad (42)$$

$$\vec{g}^{\,M}(t,z) = \vec{z}_0 g^{\,M}(t,z) = \frac{\vec{z}_0\tilde{n}^2 E_0^2}{c^2 Z}\cos(\omega t - \beta(\omega)z)\cos(\omega t - \beta(\omega)z - \varphi)\exp(-2\alpha(\omega)z)\ . \qquad (43)$$

Now we introduce one more definition, which more closer to Minkowski one: $\vec{\tilde{g}} = \vec{z}_0\tilde{g}^{\,M}$, where $\tilde{g}^{\,M} = S/v_p^2 = n'^2(\omega)g^{\,A}$. At $\sigma = 0$ we have $g^{\,M} = \tilde{g}^{\,M}$, $\alpha = 0$, $\beta = \omega\sqrt{\varepsilon'\mu}/c$, and in this case the momentum on average is not transmitted to matter (with the exception of reflections from the boundaries), because the Lorentz force and the electric polarization current density $\vec{J}_P^{\,e} = \varepsilon_0(\varepsilon'-1)\partial\vec{E}/\partial t$ are phase-shifted on $\pi/2$, and the momentum propagates with phase velocity $v_i = v_p = c/\tilde{n}$ [29]. It also is related to the Lorentz force $\varepsilon_0\varepsilon'\vec{E}\times\vec{J}_P^{\,m}$ for magnetic polarization current density $\vec{J}_P^{\,m} = \mu_0(\mu-1)\partial\vec{H}/\partial t$.

In dissipative media due to photon absorption the momentum is transferred to matter, its flow is directed along *z*, therefore the balance equation for momentum density vector may be written as [19,29]

$$-\partial_t[g(t,z) + g_\sigma(t,z)] = \partial_z[v_i(t,z)g(t,z)] = v_i(t,z)\partial_z g(t,z), \qquad (44)$$

where the transmitted to matter momentum $g_\sigma(t,z)$ has been designated. By virtue of medium homogeneity the velocity v_i here is not dependent on *z*. For any of three forms one has the functional dependence $g(t,z) = G(\omega t - \beta z)\exp(-2\alpha z)$. The velocity of momentum

change transferred to matter for thickness layer dz with unite square is the difference between the flows of g through the sections z and $z+dz$, so

$$\partial_t g_\sigma(t,z) = 2\alpha v_i(t,z) g(t,z) \ . \tag{45}$$

The value (45) is equal to field pressure on the unite thickness layer. In order to get the pressure on some finite thickness layer one must integrate the expression (45) over its coordinates. If the layer is infinite then all momentum is transferred to the matter. Substituting the (45) into (44), we get that the momentum transfer velocity is constant and equal to phase velocity:

$$v_i(t,z) = -\partial_t g(t,z) / \left[\partial_z g(t,z) + 2\alpha(\omega) g(t,z)\right] = \omega / \beta(\omega) = v_p(\omega) \ . \tag{46}$$

Here we do not concretize the momentum form. One may get this relation from the following reasoning. The Maxwell equations are taken form (24). Hence the balance equation is

$$\partial_z \left[\varepsilon_0 \varepsilon' E^2 / 2 + \mu_0 \mu H^2 / 2\right] = -\partial_t g^M - \mu_0 \mu \sigma E H \ . \tag{47}$$

As $\partial S / \partial t = -v_p \left(\partial S / \partial z + 2\alpha S\right)$, that the unique form $g = \eta S / c^2$ which compensates the acting on the charges Lorentz force $\mu_0 \mu \sigma S$ is obtaining under the form $\widetilde{g}^M$. Indeed, we demand the fulfillment of balance equation $2\alpha\eta S v_p / c^2 = \mu_0 \mu \sigma S$. This balance is written for matter charges, from which $\eta = c^2 \omega \mu_0 \mu \sigma \beta / \left(2\alpha\omega^2\right) = c^2 / v_p^2$. Therefore we do the replacement $\partial g^M / \partial t = \partial \widetilde{g}^M / \partial t + \widetilde{f}$ in the right part of (47), where $\widetilde{f} = c^{-2}\left(\widetilde{n}^2 - n'^2\right)\partial S / \partial t$. The average energy density in (47) has the presentation $\langle u \rangle = \langle S \rangle / v_p = E_0^2 \cos(\varphi) / \left(2 Z v_p\right) = E_0^2 / \left(2 v_p^2 \mu_0 \mu\right)$. In a similar way for the third momentum form one has $\left\langle \widetilde{g}^M \right\rangle = n'^2 \langle S \rangle / c^2 = n'^2 E_0^2 / \left(2 v_p \mu_0 \mu\right)$. Therefore the transfer momentum velocity for $\widetilde{g}^M$ is

$$\widetilde{v}_i^M = \langle u \rangle / \left\langle \widetilde{g}^M \right\rangle = c^2 / \left(n'^2 v_p\right) = v_p(\omega) . \tag{48}$$

The additional term $\widetilde{f}$ appears in this case in balance equation. It is similar to Abraham force and acts on the medium. But the appearance of indicated force (as and Abraham force) must not lead to any objections on balance equations nonfulfillment, as $\langle \partial S / \partial t \rangle = 0$ for any phase shift φ. Therefore, the mentioned force, as well as the Abraham force, does not transmit the momentum to the matter on average. In non-conducting medium $\widetilde{f} = 0$, $\widetilde{g}^M = g^M$, and the energy and the momentum both propagate with the phase frequency independent velocity $v_p = \omega / \beta = c / \widetilde{n}$. Carrying the divergent part of $\widetilde{g}^M$ into the left part of equality (47), one get the another balance equation $\left\langle u - S / v_p \right\rangle = 0$ which is the identity. The residuary before averaging term in its right part $\widetilde{f}$ is the force acting on the matter but is not transferring the momentum on average over a period. Notice, that usage of traditional Minkowski momentum density does not lead to any physical clear expression for its velocity in conductive medium. But if we have the localized initial electric current and $\sigma = 0$, then the usage $\vec{g}^M$ is more convenient than $\vec{g}^A$ and leads to the full momentum transportation with the velocity $v_i = v_p = v_g = v_e = c / \widetilde{n}$, that is trivial as there is no dispersion. Essentially thus the EMT is traditionally introduced for media [3–5].

In big number of papers the "correct" momentum density is defined as Abraham one $g^A(t,z) = S(t,z) / c^2$ [3–5]. As the result, the additional volume Abraham force $f^A = c^{-2}\left(\varepsilon'\mu - 1\right)\partial S / \partial t$ arises. Such definitions is supposed more rigorous, though the Minkowski definition is frequently more convenient for continuum [3,5] and corresponds to

experimental data for light pressure. The balance equation (47) may be written in such manner

$$\partial_z\left[\varepsilon_0\varepsilon' E^2/2+\mu_0\mu H^2/2\right]=-\partial_t g^A-f^A-\mu_0\mu\sigma S\ . \tag{49}$$

Now the momentum density change per second acting on the matter is $\left(2\alpha\omega\varepsilon'\mu/\beta\right)S/c^2$. This is the first and the second terms in the right part of (49) result of contribution. But it does not compensate the last term as $\widetilde{n}\neq n'$. Evidently, the usage of first or second terms in the right part of (49) separately always does not lead to any such compensation. Using the value g^A as the momentum density one does not succeed to get any rational determination of its rate of movement which is not exceeding *c* for the dissipative media. The exception is the nonconductive nondispersive medium with $\widetilde{n}=n'$. As $c^{-2}\left(\varepsilon'\mu-1\right)\partial_t S=-v_p c^{-2}\left(\varepsilon'\mu-1\right)\partial_z S$, then, introducting the divergent part of f^A into the flow density, we have

$$v_i^A=\frac{\left\langle u-v_p\left(\varepsilon'\mu-1\right)S/c^2\right\rangle}{\left\langle g^A\right\rangle}=c^2\left[\left\langle S\right\rangle/v_p-v_p\left(\varepsilon'\mu-1\right)\left\langle S\right\rangle/c^2\right]/\left\langle S\right\rangle=c\widetilde{n}-v_p\left(\widetilde{n}^2-1\right)=v_p=c/\widetilde{n}\ .$$

Consequently, in this case the filed momentum also is carrying with the phase velocity which is smaller than velocity of light and coincides with the energy velocity in nondispersive medium.

Let consider the momentum transferred to matter per unit time (second). For this we write the balance as

$$\partial_z\left[\varepsilon_0 E^2/2+\mu_0 H^2/2\right]=-\partial_t g^A-\mu_0\mu\sigma S+\mu_0\left(\mu-1\right)\sigma S-c^2\left(\varepsilon'-1\right)H\partial_t E-c^2\left(\mu-1\right)E\partial_t H\ . \tag{50}$$

This equation also has been obtained from the Maxwell equations when the medium is taken into account as the electric $J_{Px}^e=\varepsilon_0\left(\varepsilon'-1\right)\partial_t E$ and magnetic $J_{Py}^m=\mu_0\left(\mu-1\right)\partial_t H$ polarization current densities. This approach more peculiar to microscopic electrodynamics [7,8], but for all of that the description is more complicated. It is well known, that the Abraham force does not fully characterize the effect on the matter [5]. The three last terms in the (50) correctly reflect such effect. Namely, $f_\sigma^L=\mu_0\mu\sigma S=\sigma EB$ is the Lorentz force acting on conductivity current, $f_{Pe}^L=c^{-2}\left(\varepsilon'-1\right)H\partial_t E=J_{Px}^e B$ is the Lorentz force acting on electric polarization current of medium, and $f_{Pm}^L=c^{-2}\left(\mu-1\right)E\partial_t H=D_x J_{Py}^m$ is the Lorentz force acting on the magnetic one (here $B=\mu_0\mu H$). For the first force we had obtain the following averaged meaning $\left\langle f_\sigma^L\right\rangle=\mu_0\mu\sigma\left\langle S\right\rangle=\sigma E_0^2\beta/2\omega=n'\sigma E_0^2/\left(2c\right)$. For the second and third ones we accordingly have $\left\langle f_{Pe}^L\right\rangle=-\left(\varepsilon'-1\right)E_0^2\sigma/\left(4n'c\right)$ and $\left\langle f_{Pm}^L\right\rangle=\left(\mu-1\right)E_0^2\sigma/\left(4n'c\right)$. As usually $n'>>1$, the first term plays the determinative role in momentum transfer to mobile charges. They, scattering on the molecules and atoms of matter, transmit the momentum. The energy $S=v_p u$ is the one transferring per unit time through unit surface. The lost in unit volume power here is $\sigma E_0^2\cos^2\left(\omega t-\beta z\right)$, and its averaged over the period value is $\sigma E_0^2/2$. On the other hand, this value may be determined by such a way: $\left\langle\partial u_\sigma/\partial t\right\rangle=v_p^{-1}\left\langle\partial S_\sigma/\partial t\right\rangle$, that is equal to the expression $2\alpha\left\langle S\right\rangle=\alpha E_0^2\cos\left(\varphi\right)/Z=\sigma E_0^2/2$. The lost energy corresponds to the transmitted momentum $\left\langle\partial g_\sigma/\partial t\right\rangle$. For the three momentum definitions we have correspondingly:

$$\left\langle\partial_t g_\sigma^M\right\rangle=2\alpha\widetilde{n}^2 v_p E_0^2\cos\left(\varphi\right)/\left(c^2 Z\right)=\sigma\widetilde{n}^2 E_0^2 v_p/\left(2c^2\right),$$

$$\left\langle \partial_t g_\sigma^A \right\rangle = 2\alpha v_p E_0^2 \cos(\varphi)/\left(c^2 Z\right) = \sigma\widetilde{n}^2 E_0^2 v_p /\left(2c^2\right),$$
$$\left\langle \partial_t \widetilde{g}_\sigma^M \right\rangle = 2\alpha n'^2 v_p E_0^2 \cos(\varphi)/\left(c^2 Z\right) = \sigma E_0^2 /\left(2v_p\right).$$

As the free charges satisfy the equation of motion $N\dot{e}x(t) = \sigma E_0 \cos(\omega t - \beta(\omega)z)\exp(-\alpha(\omega)z)$, it may come into view that the matter has the oscillatory *x*-component of momentum. But this is not so. As long as we have assumed the matter is electrically neutral, there are the charges, which is deposing into opposite side and they have the opposite momentum (for example, in metal this is the lattice). Such question may arise: from what the filed has the momentum when it is exciting by dipole or by dipole system (by the dipole plane in our case) as soon as the sources had not them? Here the answer is trivial: each source excites two waves in opposite positive and negative directions of *z*–axis with opposite momentums having the equal modules.

It must be mark that the elementary momentum filed quanta are transferred in matter by photons between the acts of its scattering on the particles with the velocity of light *c*. The momentum transportation with phase velocity is the collective effect of these elementary acts with taking into account the phase delays and the interferences. Formally the taking into consideration of matter influence may be carried out by polarization currents introduction. For nondissipative media (i.e. for dispersionless ε' and μ) the phase shift between the filed is $\pi/2$, and the momentum in not transferable by this currents to matter. It is transferring only by conductivity current. The balance equation (50) (and the analogous one for power) with the density $u_0 = \left(\varepsilon_0 E^2 + \mu_0 H^2\right)/2$ is also not convenient for the determination of the transfer phenomena in media.

The balance relations with incident and polarization currents introduction are highly productive under the nonstationary excitation [16]. In this case the energy accumulated in any volume and the momentum is dependent on filed creation process prehistory. For example, in plasma one must consider the energy and momentum of field-matter system. For plasma it means the taking into consideration of charge particles kinetic oscillation energy and its momentum. Solving the motion equations for filed-matter system, we can in principle define the EMT of field and matter in any time and the correspondingly the values $\vec{v}_e(\vec{r},t)$ and $\vec{v}_i(\vec{r},t)$. In macroscopic electrodynamics of continuum the averaging over the physical infinitesimal volume (or homogenization) leads to material conditions which are the analogue of motion equations. For example, we write the one-dimensional nonstationary equations (24) in homogeneous medium with taking only the time (frequency) dispersion:

$$-\partial H/\partial z = \partial D/\partial t + J^e\ , \qquad -\partial E/\partial z = \partial B/\partial t + J^m\ . \tag{51}$$

All values in (51) are the functions of time and one coordinate *z* that simplifies the analysis. The dispersive material conditions will be taking in the Landau-Lifshitz forms

$$D(z,t) = \varepsilon_0 \int_0^t \varepsilon(z,t-t')E(z,t')dt'\ , \qquad B(z,t) = \mu_0 \int_0^t \mu(z,t-t')H(z,t')dt'\ . \tag{52}$$

Without lack of generality we will consider that the field was absent at $t = 0$. If the sources are located at the plane $z = 0$, then for the time *t* the field will be located within the bounds $|z| \le ct$. The conductivity corresponds to pole at zero in the spectral complex permittivity

$$\varepsilon(z,\omega) = \int_{-\infty}^{\infty} \varepsilon(z,t)\exp(-j\omega t)dt = \int_0^t \varepsilon(z,t')\exp(-j\omega t')dt'\ .$$

Here we have $\varepsilon(z,t) = 0$ at $t < 0$. The plasma model is more convenient here. The wave in plasma has the Landau attenuation, and this pole may be eliminated. Writing the balance equations for the momentum, one can get such form

$$\partial_z u_0(z,t) = -\left[\mu_0 H \partial_t D(z,t) + \varepsilon_0 E \partial_t B(z,t)\right] - \mu_0 H J^e - \varepsilon_0 E J^m \ . \qquad (53)$$

The last two terms here create the field and matter momentum. They correspond to Lorentz force in vacuum but not in matter. Meaning that in the square bracket must stay the value like the $\partial_t g(z,t)$, we get after the integration the following relation

$$g(z,t) = \partial_{t'}^{-1}\left(\varepsilon_0 E(z,t')\partial_{t'} B(z,t') + \mu_0 H(z,t')\partial_{t'} D(z,t')\right) =$$
$$= \varepsilon_0 \varepsilon(z,0) E(z,t) + \mu_0 \mu(z,0) H(z,0) + \partial_{t'}^{-1}\left(\varepsilon_0 \partial_t \varepsilon(z,t-t') E(z,t') + \mu_0 \partial_t \mu(z,t-t') H(z,t')\right) = \qquad (54)$$
$$= \mu_0 H(z,t) D(z,t) + \varepsilon_0 E(z,t) B(z,t) - \partial_{t'}^{-1}\left(\mu_0 \partial_{t'} H(z,t') D(z,t') + \varepsilon_0 \partial_{t'} E(z,t') B_y(z,t')\right).$$

From this follows that the *z*-component (54) at time *t* depends not only on filed values in given present moment, but on all their previous values. For homogeneous plasma we have $\mu(z,t) = \delta(t)$, $\varepsilon(z,t) = \delta(t) + \left(\omega_p^2 / \omega_c\right)\left(1 - \exp(-\omega_c t)\right)\exp(-\omega_L t)$. The last exponent here determines the Landau attenuation, and in the final results one may take the limit $\omega_L \to 0$. Then $\varepsilon(\omega) = 1 + \omega_p^2 / \left(\omega^2 - j\omega\omega_c\right)$. Setting the electrical current density as $J^e = I\left[1 - \exp(-t/\tau)\right]\sin(\omega t)$, the equation (51) may be solved and the density (54) determined at grate times *t*, when the process is almost stationary. But the relation (54) does not agree with momentum in continuum. For vacuum it is trivial: $g = g^A = S / c^2$. The value more agreed with continuum is the Minkowski density $g = g^M = DB$. The Corresponding to it and to equations (51) momentum balance equation has the form

$$\partial_z \Sigma + \partial_t g^M = -B_y J_x^e - D_x J_y^m \ . \qquad (55)$$

Here $\partial_z \Sigma = B\partial_z H + D\partial_z E$, $g^M = DB$, Σ is the *z*-direction momentum flow density, and the real Lorentz forces are in the right part of (55) taking into account the collective influence of all charges motion in the continuum. Evidently, the relation (55) more acceptably for continuum as without the dispersion i.e. at the material relations $D = \varepsilon_0 \varepsilon E$, $B = \mu_0 \mu H$ we have $g^M = \varepsilon\mu S / c^2$, $\Sigma = u = \left(\varepsilon_0 \varepsilon E^2 + \mu\mu H^2\right)/2$, $v_i^M = c / \sqrt{\varepsilon\mu} = v_p$. Therefore we have got the generalization of Minkowski momentum density. The difficulty for nonstationary case is in calculation of $g^M = DB$ and Σ that demands the knowledge of all process prehistory, and the flow Σ must be found as the solution of differential equation yet. For the one-dimensional case of plane wave diffraction on the plate it is sufficient to fulfill the integration over *z*. The Abraham force introduction in (55) (as the makeweight to Abraham momentum density in order actually to get the Minkowski one) only confuses the consideration. Let get the corresponding to (51) power balance equation $p = \partial_t w = \partial_t (u + q)$: $\partial_t w + \partial_z S = -E J_x^e - H J_y^m$. Here $q(z,t)$ is the dissipated source work or heat which always may be calculated. For the simplest case of conducting medium it is $q(z,t) = \sigma \partial_t^{-1}\left(E^2(z,t')\right)$. From here the field-matter energy density follows as:

$$u(z,t) = \int_0^t \left(E(z,t')\partial_{t'} D(z,t') + H(z,t')\partial_{t'} B(z,t')\right)dt' - q(z,t) . \qquad (56)$$

7. Conclusions

The presentations for plane monochromatic wave energy and momentum densities have been obtained for the dispersion determined only by conductivity. It had been shown that in this case the transport energy and momentum velocities are identical and equal to phase one which always is smaller than the velocity of light in vacuum, whereas the group velocity may exceed the light velocity. The several forms of momentum densities have been considered and it had been shown that the Minkowski form in this case is more rather

than the Abraham one. The result for mentioned energy transportation velocity has been received by several independent ways, in particular, by using the dispersion law for gas of oscillators. These conclusions are generalized for media with several resonant frequencies and also for the inner field availability. The essential here is the calculation of polarizability with the use of particle motion equation which has the first order. In this case the matter potential energy is not accumulated, and its characteristic oscillations (or kinetic energy) are absent. This result may be generalized for conducting polar dielectrics such as see or water, which contains the conductivity ions. The retardation in ideally distillated water changes from 9 about to 1 (without the influence of infra-red and ultra-violet resonances), and the energy velocity coincides with phase one. The retardation and loss coefficients in water with ions for ultra low frequencies tend to infinity. In see water ($\sigma = 4$ С/м) at the frequency about 900 MHz the displacement current is equal to conductivity current, and for the essentially low frequencies the water is likewise a metal. The energy and momentum transportation velocities knowledge is important for communication. For example, the communication with submarines is realized at extremely low frequency, and the signal hare is transmitted with about the phase velocity. The group velocity in see water is approximately in two times greater. For example, the pulse with the carrier frequency 1 kHz and with the duration 2 ms (or two periods) reaches the submarine located at the depth of 100 m during 2 ms and its envelope maximum is detected at that time (without the detector delay time consideration), whereas the strongly diffused and degraded front of pulse belled in several times comes during 1 ms must and is not be detected [38].

The question about EMT and corresponding densities forms in electromagnetics of continuum which is known as Abraham-Minkowski contravention is still open and discussion (see additionally [48–53]). There are different papers with model examples pro and contra of appointed EMT forms which often contain the inaccuracies. We have not opportunity to analyze it in this paper (see [12]). In overwhelming number of works the static material conditions $\vec{D} = \varepsilon_0 \varepsilon \vec{E}$, $\vec{B} = \mu_0 \mu \vec{H}$ are used for the analysis of this question. The main objection against the Minkowski EMT states in fact that it is asymmetrical and the momentum conservation law has not fulfilled for it [4]. The component T_{44} for both EMT forms is defined as in static, i.e. in form $T_{44} = u = \left(\varepsilon_0 \varepsilon \vec{E}^2 + \mu_0 \mu \vec{H}^2\right)/2$, whereas there is the interaction between filed and matter in dynamics, i.e. one must use the expressions like (56) [15,16]. It is not clear why the energy density in media should be depended on its parameters, but the momentum density $\vec{g}^A$ is nondependent like as in vacuum. It is clear, that the filed-matter interaction may leads to full EMT asymmetry [13]. It had been shown above that the momentum density depends on the process prehistory and is determined by it. Evidently, we can speak about the concrete EMT form as algebraic relations only for harmonic process in the media with definite dispersion laws when this process is gotten as the limit transition from quasi-monochromatic one and when the averaged over the period components of $\hat{T}$ came out to stationary levels. The general full filed-matter system EMT does not construct for arbitrary (time and spatial) dispersion and the question of its symmetry is open yet. If the field and matter tensors are symmetrical separately, it is means that there in no field-matter interaction from the viewpoint of energy and momentum exchanges.

As the example we now will demonstrate the Minkowski momentum conservation law fulfillment. Let the plane quasi-monochromatic hence long wave pocket with length size l with the rectangular envelope and carrier frequency ω (the pulse in sense of nonstationary wave) at the moment $t_0 = 0$ had approached to the boundary of layer with the thickness d and the permittivity and permeability ε, μ. The quasi-stationary means that $l >> \lambda_0 = 2\pi / k_0 = 2\pi c / \omega$. It is required in order to one may use the monochromatic values.

The wave-length in the layer is $\lambda = \lambda_0 / n$, $n = \sqrt{\varepsilon\mu}$. Let for simplicity the thickness d is matched so that at the time t_1 all wave train came into the layer and fully filled it. As the wave pocket velocity is $v_p = c/n$, we get $t_1 = d/v_p = nd/c$. In order to it will be, we must impose $l = nd$. We select the big length l so that it contains the integer number M of wave-lengths: $l = M\lambda_0$, $M >> 1$, and for this $d/\lambda = M$. There is the reflected wave also. At the time t_1 it is located in the same region $-l \le z \le 0$ as the incident wave at the time moment t_0. The electric field reflection coefficient for normal dip plane monochromatic wave is $R = (\rho_0 - 1)/(\rho_0 + 1)$, where $\rho_0 = \sqrt{\mu/\varepsilon}$ is the normalized to Z_0 impedance, and for the transmission coefficient we have $T = 2\rho_0/(\rho_0 + 1)$. The incident wave has the form (41) under $\alpha = 0$ and $\varphi = 0$. Further we will consider the balance for unit square and averaged momentum densities. At the time t_0 the field had the momentum $G_0^M = g^M l = \langle S_0 \rangle l / c^2$ in z-direction, and the plate momentum was zero. At first let consider the ideal case of matched (stealth) plate: $\varepsilon = \mu$. In this case the reflection is absent, and the momentum is not transferred to the plate. The factor ½ arises at the amplitudes after the integration over z or averaging over the period. Correspondingly $g^M = Z_0 n^2 E_0^2 / (2c^2) = \langle u \rangle n / c$, where $\langle u \rangle = \varepsilon_0 \varepsilon E_0^2 / 2$. At the moment $t_1 = l/c$ of full plate filling the field momentum is equal to $G_1^M = g^M d = dn^2 \langle S_1 \rangle / c^2 = dn \langle u_1 \rangle / c = nG_0^M$. After the wave train completely went out from the plate, i.e. in the time $t_2 = 2t_1$ and after we again have $G_1^M = G_0^M$. The Minkowski momentum as though is not conserved as the plate is motionless. Such "nonconservation of Minkowski momentum" raises in the work [4] as the basic argument against Minkowski and for Abraham momentum. Let see what will be with the Abraham momentum. At t_0 we have: $G_0^A = G_0^M = \langle S_0 \rangle l / c^2 = \langle u_0 \rangle l / c$, and at t_1 accordingly $G_1^A = \langle S_1 \rangle d / c^2 = \langle u_0 \rangle d / c = G_1^M / n^2$. We have got the paradox: neither Minkowski and Abraham momentums are conserved, and under this for the first one the additional momentum arises from the beginning and then disappears, and for the second one the part of momentum disappears from the beginning and then it again appears! What is the matter? Always when any paradox arises, one must seek where he did the exchange of conceptions or illegal used theirs. In the work the similar reasoning had been proved for plate with $\mu = 1$, $n = \sqrt{\varepsilon}$, in which connection it is assumed that the plate is ideally matched by help of antireflecting coating. Though the plate can not be ideally matched, it is possible to get the sufficiently small reflection for monochromatic process by multilayered of inhomogeneous antireflecting coating. Naturally this coating must be considered. But we have the rectangular wave pockets having all frequencies, and the process is nonstationary! No doubt that the main spectral intensity is concentrated nearby the carrier frequency, but nevertheless, the plate will get any small momentum. For unreflecting plate the condition $\varepsilon = \mu$ may be fulfilled only for artificial highly dispersive media in very narrow frequency band, and in this loss case these parameters are also complex. It means that the real stealth plate also will get some momentum. We will abstract one's mind from this further. Notice, that the formula (1.13) from [4], supposedly showing the Abraham momentum conservation, in reality shows its violation: $G = e/nc$. Here e is the full wave energy. It is unified at any wave pocket position. In the vacuum we have $e = \langle u_0 \rangle l$, and in the media correspondingly $e = \langle u \rangle d$. And as the train is squeezed in last case in n times, $\langle u \rangle = n \langle u_0 \rangle$, that one can see directly from (41). The full momentum for both cases is $G = e/c$.

So, how to resolve the paradox? It is clear, that for our case without the Lorentz force the local balance equation is

$$\partial u / \partial z = -\partial g^M / \partial t \,, \tag{57}$$

that directly follows from (41). If the local balance fulfills then the global (integral) balance fulfills also, and it must be fairly calculated. We notice for this that $u(z,t)$ has the steps $u_0(0,t)(n-1)$ at $0 \le t \le t_1$ and $u_0(d,t)(1-n)$ at $t_1 \le t \le t_2$. Correspondingly there are the delta-function terms $u_0(0,t)(n-1)\delta(z)$ and $U_0(d,t)(1-n)\delta(z-d)$ in (57) at the indicated time moments. The momentum at the time t_1 must be calculated in the following way:

$$G_1^M = -\int_0^{t_1}\left[\int_0^d u(z,t)dz + u_0(0,t)(n-1)\right]dt = nG_0^M - (n-1)G_0^M = G_0^M .$$

The mentioned conclusion about G^M "nonconservation" in [4] has been made on the mass center velocity constancy for the system: the motionless plate – moving electromagnetic wave train (photon). The first one has not a velocity, and the second one has not a mass [40,41]! Meantime the "mass of photon" has been introduced in [4] from the relation $m = e/c^2$ and also its momentum has been designated as $G = mv_p = e/(cn)$! Speaking about the wave pocket or photon in the medium it is necessary take into account that the energy density increases in *n* times, consequently the number of photons rises so, and the energy transportation velocity declines. This is means that there are the both direction photons in the plate moving with the velocity *c* between the elementary interaction acts with the particles, and the photon with the momentum $n\hbar\omega/c$ is, as a matter of fact, the quasi-photon [3]. The full interference result for macroscopic wave with big energy and many photons is the consequence of cancellation theorem, according to which the wave in medium goes in direct direction with phase velocity. For general case of reflection from non-absorptive plate at $t > t_2$ one has the energy balance: $1 = |R|^2 + |T|^2$. Here $R = (\rho_0^2 - 1)\tan(\theta)/((\rho_0^2 + 1)\tan(\theta) - 2j\rho_0)$ is the full reflection coefficient and $T = (\cos(\theta) + j\sin(\theta)(Z^2 + 1)/(2Z))^{-1}$ is the transmission coefficient, $\theta = \gamma l = 2\pi l / \lambda_0$, and so the reflected momentum is $G_r^M = e|R|^2 / c$, and transmitted one – correspondingly $G_t^M = e|T|^2 / c$, i.e. the plate gets the momentum $e(1 + |R|^2 - |T|^2)/c$. If we take $R = 0$, then $T = 1$, and the plate is immovable. If ever $\sigma \ne 0$, then the value γ is complex, and the energy balance is such: $|R|^2 + |T|^2 < 1$. For this some part of momentum in the any layer is transferred to it. When the $|R|$ is extremely small and the thickness is extremely large then the plate gets all momentum. At the limit $\sigma \to \infty$ we have $R \to -1$, $T \to 0$, and the plate gets the doubled momentum. We again have applied the stationary formulas to quasi-stationary processes instead of the solution more complicated nonstationary problem that is justified for long wave train. For the stationary case it means to say only about the light pressure $\langle u \rangle (1 + |R|^2 - |T|^2)/c$ or about the transferred to matter per the second momentum, and at the same time the formulas already are exact. As regards to momentum G^A, that it is conserved only if the additional momentum conditioned by Abraham force is added to it. The mention force for nondispersive medium arises only at the train entrance into the plate and at the exit from it [5]. Exactly, the necessity of adding G^A with some momentum to provide the conservation law is the main argument against. The other such argument against the Abraham momentum in the media is based on the fact that the Abraham force is not precisely equal to Lorentz forces acting on polarization medium current.

For example let consider two atoms with the masses m_1 and m_2 motionless in laboratory system at the times $t<0$. The system energy here is $e=e_1+e_2$, $e_1=m_1c^2$, $e_2=m_2c^2$. The first atom is excited with the excitation energy $\hbar\omega$ and is located at $z=0$, and the second one – at the point $(0,0,z)$, $z>0$. The excited atom is immovable (as quantum particle) from the viewpoint that its wave function is time-independent (precisely has the factor $\exp(-ie_1t/\hbar)$, $i=-j$). And at the same time the probability to disclose the atom at $z=0$ is maximal, and the indeterminancy principle here fulfills: $\Delta z\Delta p_z\geq\hbar/2$. All of these are regarded to the second atom. As there is no interaction, the full wave function is the product of atomic functions and is the time-independent until the time $t_0=0$. In the time moment $t_0=0$ the first atom radiates the photon with the energy $\hbar\omega$ and the momentum $\vec{p}=\vec{z}_0p=\vec{z}_0\hbar\omega/c$. The atom gets the momentum $-\vec{p}$ and moves to the left, that means the dependence of its wave function on the time. This wave function gets the wave pocket presentation as the eigen-function of momentum operator. The wave pocket displaces to the left with the velocity satisfy the relation $(v_1/c)/\sqrt{1-(v_1/c)^2}=p/(m_1'c)$, and under this $m_1'=\sqrt{(e_1-\hbar\omega)^2/c^4-\vec{p}^2/c^2}=m_1\sqrt{1-2p/(m_1c)}$ is the mass change due to interaction. The atom excitation energy $\hbar\omega$ plays the role of incident inner nonelectromagnetic energy which creates the field. Let at the time t_1 the photon at the point $z=ct_1$ is absorbed by other atom. This atom gets the momentum $\vec{p}$ and turns into excited state with the energy $e_2+\hbar\omega=m_2'c^2/\sqrt{1-(v_2/c)^2}$. And so its mass $m_2'=m_2\sqrt{1+2p/(m_2c)}$ is changed. The atom velocity is also determined by the relation $(v_2/c)/\sqrt{1-(v_2/c)^2}=p/(m_2'c)$. Until the time t_0 the system of noninteracting atoms had the null momentum, mass m_1+m_2 and the energy $(m_1+m_2)c^2$. In the period $t_0<t<t_1$ the mass is $m_1'+m_2$ and smaller than the initial, and the photon mass is zero, but the full system filed-matte mass does not change and is equal to $m+m_0$, as full system momentum is zero. The energy and momentum are conserved here, and the photon position is non-designated. The photon is localized and the filed disappears at the time of interaction at the point *z*. the all system mass again is equal m_1+m_2 and is conserved during all time. The full momentum is also conserved and is zero. The full energy is $e=m_1'c^2/\sqrt{1-(v_1/c)^2}+m_2'c^2/\sqrt{1-(v_2/c)^2}$ and conserved also. The total atomic mass is equal to $m_1'+m_2'<m_1+m_2$. There is the mass defect due to the fact that the atoms have got the opposite momentums. The full energy, momentum and mass values of this closed system had not changed. If $2p/(m_ic)<<1$, $i=1,2$, then one may use the decomposition on small parameter. Evidently, to a first approximation there is no mass defect. This qualitative example based on nonstationary interaction replacement by stationary processes with two point interactions is given here in order to show the necessity of taking into account the incident sources in the balance, including if the field has the angular momentum. It is also useful in connection with the example from [4]. The wave propagation in the continuum on microscopic level is based on similar numerous interaction acts, and the transport velocity depends on atomic life times, i.e. on scattering character: how much it is elastic. There is the preferred flow of photons in energy propagation direction in such waves always, but the backward photons are possible. The macroscopic level of consideration shows that the wave creates the polarization currents which are supporting the wave from the other hand. It is follows from the stated that the photon exchange processes are always nonstationary

and at least quasi-stationary. Therefore the monochromatic wave is in a certain sense the convenient abstraction which never experimentally in principle may be obtained.

In conclusion it is useful to notice that the concrete dispersion laws of real media are highly complicated. It is necessary to use the inner molecular or crystalline structure, the inner molecular field, several eigen-resonant frequencies, the spatial dispersion (if it exists), and also the nonstationary approach. It means that the EMT for filed-matter is not determined only by the field values in the present time, i.e. the process prehistory must be taken into account. If there are no any sources in the volume, one must consider such prehistory from the time of entering the before created field into this volume. Therefore the momentum density should be taken in the form $\vec{g}^M = \vec{D} \times \vec{B}$ with the calculation of (1), and the energy density must be calculated using the relations like (56) [15,16]. The full field-matter EMT for this is asymmetrical. For dispersionless case it turns to the Minkowski EMT. The value $\partial_z u$ determined by the relation (56) does not give the momentum flow density for plane wave. The usage only the material conditions in general case allows one to separate the filed and matter momentums and its flows. Such separation for energy density flow apparently has no place without of solution of motion equations. It is connected with the fact that the polarization current power density in right part of (9), which is containing the time derivatives, is rather corresponded to full power density, than is the divergence of any vector. What is more the mentioned vector is ambiguously determined. Besides the system interaction energy in general case is not separable on field energy and matter energy. The main remaining objection contra the Minkowski EMT consists of its dissymmetry. The demand of symmetry arises from the requirement of EMT uniqueness and the condition that the angular momentum tensor must be determined via the EMT using the standard formulas. In this case the connection of momentum components with EMT components is such like in vacuum ([52], page 107). It is obvious that such EMT is the filed tensor in vacuum or Abraham tensor. The EMT for media in our nonstationary excitation case is determined unambiguously without any additional condition. The angular momentum for filed-matter system must be determined separately. Here it is necessary to take into account the following facts. 1) Had the matter any angular momentum before the creation of field, or not. 2) Had any angular momentum been transferred to the incident sources of not in the process of filed generation.

The obtained EMT expressions depend on material condition. If one takes other material conditions, for example, the Casimir conditions [18], he gets other ones. Recently the question about the EMT and corresponding densities (and velocities) arises for artificial media (metamaterials) with spatial dispersion including the bianisotropic and left-handed media (see for example подобный such lame attempt in [53]). Similar media have the very complicated rigorous models. However in several cases the dispersion laws for certain frequency bands may be described by more simple relations or models that had been used in this paper.

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MICROSRUCTURED OPTICAL FIBER WITH ULTRA-LOW DISPERSION

A.G. Rozhnev, *Member IEEE*, A.V. Sadovnikov

Saratov State University, Saratov, Russia
e-mail: leosasa@mail.ru

Abstract – We report the control of dispersion in the optical fiber with photonic crystal cladding. The specific type of fiber is considered. The structure of fiber cladding is comprised of the hexagonal lattice of rods with finite wall thickness. The Finite-Element method and the Plane Wave method with a supercell modification is applied to compute the dispersion. We have demonstrated that dispersion slope and absolute value of dispersion coefficient can be controlled with a variation of geometrical parameters of fiber cladding.

1. Introduction

The optical fiber with photonic crystal cladding provide new approaches for achieving single transverse mode guiding in large core fibers. Such fiber often is named [1] microstructured fibers (MF). There are many applications of MF in fiber-optic communications, fiber lasers, nonlinear devices, high-power transmission devices, ultra-broad supercontinuum generation and broadband dispersion compensation. MF with a solid core, which has a higher average index than the microstructured cladding, can operate on the same index-guiding principle as an ordinary optical fiber. Such fiber exhibit novel dispersion properties such as ultra-flat dispersion over broad wavelength range [2]. We considered the MF with the hexagonal lattice of rods with finite wall thickness in the fiber cladding. Thereby we assumed the complicated triangle-form area in the transverse section of MF that usually appears during the manufacture of such fiber. The control of dispersion in fiber has been demonstrated by variation of geometrical parameters of fiber such as size of an air-hole defect in the fiber core and distance between holes. The effect of varying cladding parameters is estimated to find the trends of the decrease of dispersion slope and control of absolute value of dispersion coefficient and then applied to the design of an ultra-low, ultra-flattened chromatic dispersion photonic-crystal fiber.

2. Simulation description

We consider the MF that has a transverse section consisting of a hexagonal lattice of rods with finite wall thickness (Figure 1a). The number of rings of rods is assumed to be 3. The radius of the air-holes is r and the lattice pitch (the distance between air-holes) is Λ.

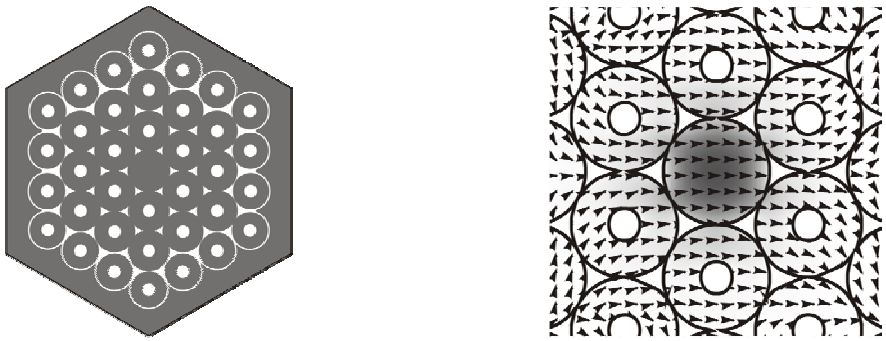

Fig. 1. a) The transverse section of fiber that has produced of rods with finite wall thickness.
b) The intensity distribution of transverse electric field for E_x-mode in the core of fiber

The numerical modeling of microstructured fiber was performed using two methods, namely finite-element method (FEM) and the plane-wave method (PWM). The former assumed the explicit frequency or wavelength dependence of the refractive index. The latter was applied with the supercell modification and the material dispersion wasn't assumed explicitly. However we summed the waveguide (geometrical) dispersion $D_w(\lambda)$ obtained by PWM and the material dispersion $D_m(\lambda)$ obtained by Sellmeier's equation [2] and the dispersion coefficient $D(\lambda)$ is then calculated as

$$D(\lambda) = D_w(\lambda) + D_m(\lambda), \quad (1)$$

where waveguide and material dispersion are calculated with $D_w(\lambda) = -\frac{\lambda}{c}\frac{d^2 n_{eff}(\lambda)}{d\lambda^2}$ and $D_m(\lambda) = -\frac{\lambda}{c}\frac{d^2 n_{Si}(\lambda)}{d\lambda^2}$ respectively, $n_{eff}(\lambda)$ is the effective refractive index of the fundamental mode in MF, $n_{Si}(\lambda)$ is the refractive index of silica, c is the speed of light in vacuum.

For the computation with finite-element method the program for calculation of eigenmodes dispersion of the transmission line with complicated transverse section [4] was applied. The plane-wave simulation was done by MPB (MIT Photonic Bands) package [5]. Simulation was conducted by these methods to investigate the effect of varying r and Λ.

The intensity distribution and polarization of electromagnetic field for fundamental mode of the fiber are shown in Figure 1b. This mode is polarized along the horizontal axis and the results for other mode that polarized along vertical axis is omitted because there is a similarity between them. The electromagnetic field energy is localized in the fiber core. It can be seen from Figure 1b, where the intensity of transverse electric field is depicted with shades of gray.

Figure 2a shows the calculated dispersion curves for wavelength range from 0.8 to 1.7 μm for different values of distance between air-holes Λ. The dispersion curve of $\Lambda = 1.84$ μm (bold curve *3* in Figure 2a) demonstrate that there is the wavelength range $0.9 < \lambda < 1.1$ μm with the small change in the dispersion coefficient D and the absolute value of D lies in the interval $0 < D < 7$ ps/(nm·km). Figure 2b demonstrates that dispersion coefficient decreases with decrease of air-hole radius r and that there is a flat region offset in the range of shorter wavelengths.

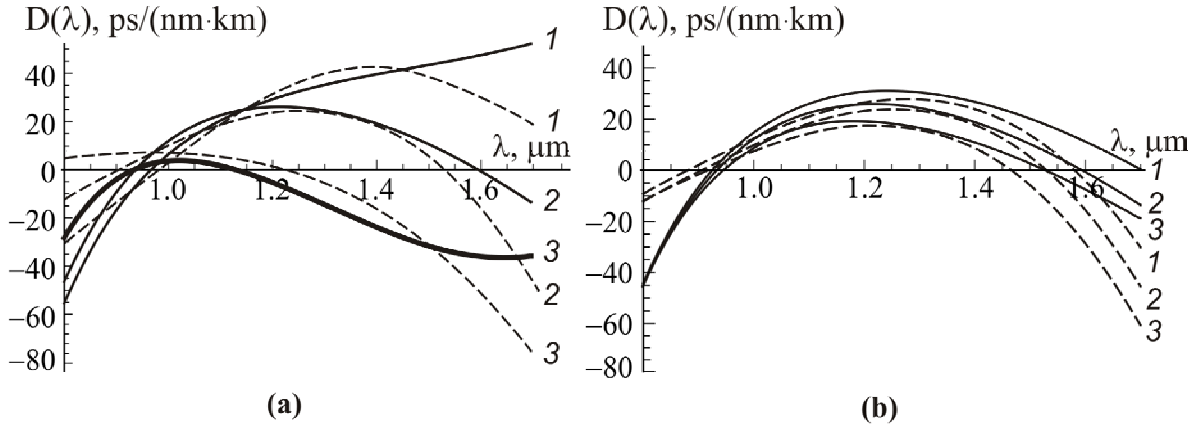

Fig. 2. Family of dispersion curves with various a) lattice pitch: *1* - $\Lambda = 3.68$ μm; *2* - $\Lambda = 2.76$ μm; *3* - $\Lambda = 1.84$ μm and with r fixed at 0.4968 μm; b) radius *1* - $r = 0.53475$ μm; *2* - $r = 0.4968$ μm; *3* - $r = 0.4485$ μm and Λ fixed at 2.76 μm. Computational results obtained with plane-wave method denoted with solid lines and finite-element method – with dashed lines

However, using a MF with all of the same air-hole diameter in the fiber cladding, it is difficult to control both the absolute value of dispersion coefficient and dispersion slope in wide wavelength range. An recent publication by Saitoh et al. [2] have reported that there is a new controlling technique of chromatic dispersion in MF. It is shown from numerical results that it is possible to design MFs with both ultra-low dispersion and ultra-flattened dispersion in a wide wavelength range. We have applied this method of dispersion controlling for our system. The hole radius of each air-hole ring in the fiber cladding increases 20% from hole ring to ring relative to the first one. Figure 3a shows the result of calculation for the system with different air-hole radii of each air-hole ring: the dispersion of the system with radius increases 20% from row to row denoted with dashed lines, for the system without air-hole radius difference the dispersion curve denoted with solid lines. The simulation was done for systems with different air-hole radius in the first ring. It was shown (curve 4 in Figure 3a) that this method can be used for control the dispersion slope of MF.

The control of chromatic dispersion in optical microstructured fiber can be done with doping the center part of the fiber with GeO_2 [6]. We have demonstarated that the increase of mol fraction of GeO_2 [6] in the silica core of our type of MF shift down the dispersion curve and it is possible therefore to design the MF with small value of dispersion coefficient in the definite wavelength range ($1.0 < \lambda < 1.05$ μm) as shown in Figure 3b (bold curve *2*).

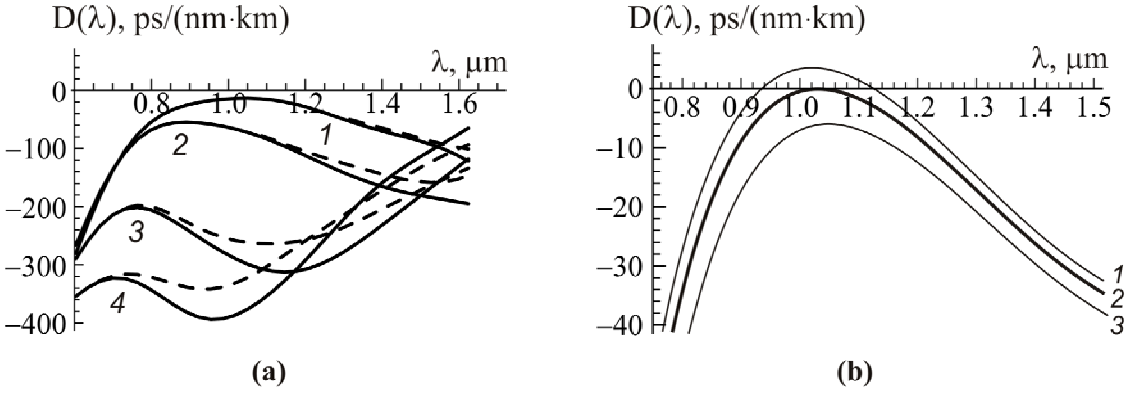

Figure 3. a) Family of dispersion curves with various air-hole radius obtained with finite-element method. 1 - $r = 0.345$ μm ; 2 - $r = 0.2875$ μm ; 3 - $r = 0.2$ μm ; 4 - $r = 0.1725$ μm . The computational result for the system with radius increase 20% from row to row denoted with dashed lines, while solid lines denotes the system without such radius increase. $\Lambda = 2.3$ μm . b) Family of dispersion curves with various mol fraction of GeO_2 obtained with plane-wave method: 1 - $x = 0$; 2 - $x = 0.056$; 3 - $x = 0.15$. $r = 0.4968$ μm , $\Lambda = 1.84$ μm

3. Conclusion

We have shown that the variation of geometrical parameters of optical microstructured fiber can modify the dispersion properties of such system. The Finite-Element method and Plane-Wave method have been applied. We have considered the structure of fiber cladding that is comprised of the triangular lattice of rods with finite wall thickness. We have demonstrated that dispersion slope and absolute value of dispersion coefficient of MFs can be controlled with a change of size of an air-hole defect in the fiber core, with variation of interval between holes and with a doping the center part of the fiber with GeO_2. Simulations and estimations shows that one can construct a system based on

microstructured-fibers with minute difference of dispersion coefficient in the operating range of wavelength and dispersion control can help to improve transmission speeds and distances.

Acknowledgments

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LINEAR ANALYSIS OF A TRAVELING WAVE TUBE AMPLIFIER WITH END REFLECTIONS

N.M. Ryskin, *Member IEEE*, A.V. Yakovlev

Saratov State University, Saratov, Russia
E-mail: RyskinNM@info.sgu.ru

Abstract – Linear theory of a traveling wave tube (TWT) amplifier with end reflections is presented along with its further generalization for the case of multi-sectional tube. The method of solution of the boundary problem using the Laplace transform is described. The results of gain calculations for the different TWT structures and different matching conditions are presented.

1. Introduction

Calculation of small signal gain is an important stage of traveling wave tube (TWT) theoretical analysis. Usually linear TWT analysis is performed by solving the dispersion equation or by the method of successive approximations [1-4]. However, most theories consider half-bounded systems when reflections of radiation from the ends are not taken into account. On the other hand, end reflections may strongly affect the TWT performance causing gain ripples and parasitic self-excitation [5,6]. In this paper, we develop the linear theory for a finite-length TWT connected with input driver source and output load. In Sec.2 we review basic equations of the linear TWT theory with the boundary conditions proposed in [7] which take into account end reflections. In Sec. 3 we describe solution of the TWT boundary value problem using the Laplace transform and generalize the theory for the case of multi-sectional tube. The method of solution is similar to [8] where TWT interaction near the cutoff was considered. In Sec. 4 results of numerical calculations of TWT gain are presented.

2. Basic Equations of Linear TWT Theory and Boundary Conditions

Let us start from the well-known of 1-D TWT linear equations [1-4] for amplification of a single-tone signal with frequency ω:

$$\frac{d^2E}{dx^2} + \beta_0^2 E = j\beta_0^3 KI\,, \tag{1}$$

$$\left(\frac{d}{dx} + j\beta_e\right)^2 I + \beta_p^2 I = \frac{j\beta_e I_0}{2V_0} E\,. \tag{2}$$

Here, E is electric field of the high-frequency wave, $\beta_0 = \omega/v_{ph}$ is the propagation constant in the "cold" structure, I_0 and V_0 are the beam DC current and voltage respectively, I is the first harmonic of the current, K is the coupling impedance, $\beta_e = \omega/v_0$, $\beta_p = \omega_p/v_0$, and ω_p is the electron plasma frequency. Equations (1), (2) should be completed with boundary conditions for the beam and the field. In particular, when the beam at the entrance of the SWS is modulated neither by velocity nor by density

$$I(0) = \frac{dI(0)}{dx} = 0\,. \tag{3}$$

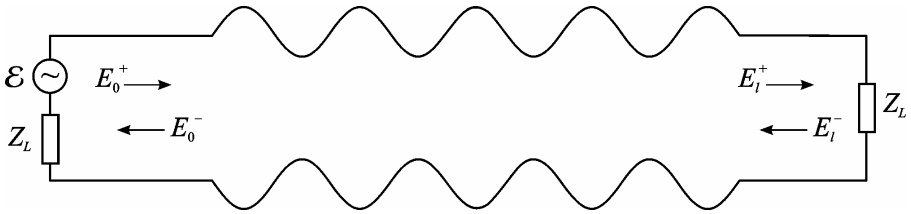

Fig. 1. Schematics of the SWS connected to input and output waveguides

Consider the finite length slow wave structure (SWS) located at $0 < x < l$ and coupled with input driver source and the output load by dispersionless waveguides on the left and right ends, respectively (Fig. 1). For that case, the following boundary conditions were proposed by A.P. Kuznetsov [7]:

$$\frac{dE}{dx} \mp j\alpha_0 E\,|_{x=0} = -2j\alpha_0 E_0^{\pm}, \tag{4}$$

$$\frac{dE}{dx} \mp j\alpha_l E\,|_{x=l} = 2j\alpha_l E_l^{\pm}. \tag{5}$$

Here, $\alpha_{0,l}$ are the parameters equal to propagation constants on the certain matching frequency. The reflection factor equals zero at that frequency and grows with moving away from the matching point. $E_0^{\pm}$ and $E_l^{\pm}$ denote amplitudes of forward and backward waves in the input and output coupling waveguide, respectively. Note that for reducing the reflections α_0 and α_l should be chosen different since the propagation constant of the forward "hot" wave differs from that of the reflected backward wave [9].

Equations (1), (2) with boundary conditions (3)-(5) define TWT linear theory boundary problem for finite length system with end reflections taken into account.

3. Solution of the TWT boundary problem

The equations of the TWT linear theory boundary problem is solved using the Laplace transform similarly to [8] where the case of interaction near cutoff was considered. Applying the Laplace transform to (1) and (2) results in

$$s^2\hat{E} - sE(0) - E'(0) + \beta_0^2\hat{E} = j\beta_0^3 K\hat{I}, \tag{6}$$

$$\left[\left(s + j\beta_e^2\right)^2 + \beta_p^2\right]\hat{I} - \left(s + 2j\beta_e\right)I(0) - I'(0) = \frac{j\beta_e I_0}{2V_0}\hat{E}. \tag{7}$$

Here, $\hat{E}$, $\hat{I}$ are images of $E(x)$, $I(x)$, respectively. One can express $\hat{I}$ using (7):

$$\hat{I} = \frac{\frac{j\beta_e I_0}{2V_0}\hat{E} + I'(0) + \left(s + 2j\beta_e\right)I(0)}{\left(s + 2j\beta_e\right)^2 + \beta_p^2}. \tag{8}$$

Substituting (8) into (6), after some manipulations we get

$$\hat{E} = \frac{\left(E'(0) + sE(0)\right)\left[\left(s + j\beta_e\right)^2 + \beta_p^2\right] + j\beta_0^3 K\left(I'(0) + \left(s + 2j\beta_e\right)I(0)\right)}{p(s)}. \tag{9}$$

Here $p(s)=\left(s^2+\beta_0^2\right)\left(\left(s+j\beta_e\right)^2+\beta_p^2\right)+2\beta_e\beta_0^3C^3$, $C^3=KI_0/4V_0$ is the Pierce gain parameter [1-4]. The forth order polynomial $p(s)$ has four complex roots $s_j=-jk_n$, $n=1,...,4$ and can be represented as

$$p(s)=(s+jk_1)(s+jk_2)(s+jk_3)(s+jk_4)=\prod_{n=1}^{4}(s+jk_n). \tag{10}$$

Substituting (10) into (9) and applying the inverse Laplace transform we obtain

$$E(x)=\sum_{n=1}^{4}c_ne^{-jk_nx}. \tag{11}$$

Here

$$c_n=\frac{-\left(E'(0)-jk_n(0)\right)\left(\left(k_n-\beta_e\right)^2-\beta_p^2\right)+j\beta_0^3K\left(I'(0)-j\left(k_n-2\beta_e\right)I(0)\right)}{p_n'}, \tag{12}$$

$$p_n'=\frac{dp(-jk_n)}{ds}=2jk_n\left[\left(k_n-\beta_e\right)^2-\beta_p^2\right]+2j\left(k_n-\beta_e\right)\left(k_n^2-\beta_0^2\right). \tag{13}$$

Similarly the following expressions for $E'(x)$, $I(x)$, $I'(x)$ can be obtained:

$$E'(x)=-j\sum_{n=1}^{4}k_nc_ne^{-jk_nx}, \tag{14}$$

$$I(x)=\frac{j}{\beta_0^3K}\sum_{n=1}^{4}\left(k_n^2-\beta_0^2\right)c_ne^{-jk_nx}, \tag{15}$$

$$I'(x)=\frac{1}{\beta_0^3K}\sum_{n=1}^{4}\left(k_n^2-\beta_0^2\right)k_nc_ne^{-jk_nx}. \tag{16}$$

Thus, the variables $E(x)$, $E'(x)$, $I(x)$, $I'(x)$ are expressed through four constants $E(0)$, $E'(0)$, $I(0)$, $I'(0)$. It is convenient to represent these expressions in matrix form

$$\begin{pmatrix}E(l)\\E'(l)\\I(l)\\I'(l)\end{pmatrix}=\hat{T}_{TWT}\begin{pmatrix}E(0)\\E'(0)\\I(0)\\I'(0)\end{pmatrix}, \tag{17}$$

where, $\hat{T}_{TWT}$ is the transmission matrix of the uniform TWT section, that could be represented as

$$\hat{T}_{TWT}=\sum_{n=1}^{4}\hat{A}(k_n)\frac{e^{-jk_nx}}{p_n'}. \tag{18}$$

Here, $\hat{A}(k_n)$ is the 4×4 square matrix which elements are defined as follows

$$a_{11}(k_n)=jk_n\left[\left(k_n-\beta_e\right)^2-\beta_p^2\right],$$

$$a_{12}(k_n)=\beta_p^2-\left(k_n-\beta_e\right)^2,$$

$$a_{13}(k_n)=\beta_0^3K\left(k_n-2\beta_e\right),$$

$$a_{14}(k_n)=j\beta_0^3K,$$

$$a_{2n}(k_n)=-jk_na_{1n}(k_n),\ n=1,...,4,$$

$$a_{3n}(k_n)=\frac{j\left(k_n^2-\beta_0^2\right)}{\beta_0^3K}a_{1n}(k_n),\ n=1,...,4,$$

$$a_{4n}\left(k_n\right)=\frac{k_n\left(k_n^2-\beta_0^2\right)}{\beta_0^3 K}a_{1n}\left(k_n\right),\ n=1,...,4\,.$$

One can express the unknowns $E(0)$, $E'(0)$ through $E_0^{\pm}$ using the boundary conditions (4):

$$E(0)=E_0^+-E_0^-,$$
$$E'(0)=-j\alpha_0\left(E_0^++E_0^-\right).$$

These expressions can be represented in the matrix form

$$\begin{pmatrix}E(0)\\E'(0)\end{pmatrix}=\begin{pmatrix}1 & -1\\-j\alpha_0 & -j\alpha_0\end{pmatrix}\begin{pmatrix}E_0^+\\E_0^-\end{pmatrix}. \tag{19}$$

Similarly, $E(l)$, $E'(l)$ are expressed through $E_l^{\pm}$ using (5):

$$\begin{pmatrix}E_l^+\\E_l^-\end{pmatrix}=\frac{1}{2j\alpha_l}\begin{pmatrix}-j\alpha_l & 1\\j\alpha_l & 1\end{pmatrix}\begin{pmatrix}E(l)\\E'(l)\end{pmatrix}. \tag{20}$$

Thus, the vector $\left(E_l^+,E_l^-,I(l),I'(l)\right)^T$ is expressed through the vector $\left(E_0^+,E_0^-,I(0),I'(0)\right)^T$ as

$$\begin{pmatrix}E_l^+\\E_l^-\\I(l)\\I'(l)\end{pmatrix}=\hat{T}\begin{pmatrix}E_0^+\\E_0^-\\I(0)\\I'(0)\end{pmatrix}. \tag{21}$$

Here, $\hat{T}=\hat{T}_0\hat{T}_{TWT}\hat{T}_l$ is the transmission matrix of the whole system, $\hat{T}_{0,l}$ are the transmission matrices of the transitions between the SWS and the coupling input/output waveguides:

$$\hat{T}_0=\begin{pmatrix}\begin{pmatrix}1 & -1\\-j\alpha_0 & -j\alpha_0\end{pmatrix} & 0\\0 & \hat{I}\end{pmatrix}, \tag{22}$$

$$\hat{T}_l=\begin{pmatrix}\begin{pmatrix}-\frac{1}{2} & -\frac{j}{2\alpha_l}\\\frac{1}{2} & -\frac{j}{2\alpha_l}\end{pmatrix} & 0\\0 & \hat{I}\end{pmatrix}, \tag{23}$$

where $\hat{I}$ is the 2×2 unitary matrix. In (21) we know the values of E_0^+, $I(0)$, $I'(0)$ and E_l^-. Thus one should express the unknowns E_l^+, E_0^-, $I(l)$, $I'(l)$. In particular, if the electron beam is not modulated at the input, i.e. current satisfy the boundary conditions (3), and the output waveguide is perfectly matched with the load, i.e. $E_l^-=0$, from (21) one can easily find the following equations

$$E_l^+=T_{11}E_0^++T_{12}E_0^-,$$
$$0=T_{21}E_0^++T_{22}E_0^-.$$

Solving these equations we get

$$E_0^-=\left(T_{21}/T_{22}\right)E_0^+\equiv RE_0^+, \tag{24}$$

$$E_l^+ = \left(T_{11} + RT_{12}\right) E_0^+ = \frac{T_{11}T_{22} - T_{12}T_{21}}{T_{22}} E_0^+ . \quad (25)$$

Here, R is the reflection factor from the finite length structure, T_{ij} are the elements of the transmission matrix $\hat{T}$. From (25) one can find the gain factor of the TWT amplifier

$$G = \frac{T_{11}T_{22} - T_{12}T_{21}}{T_{22}} . \quad (26)$$

Evidently, when

$$T_{22} = 0 , \quad (27)$$

the gain becomes infinite which means self-excitation of the amplifier. Analysis of the equation (27) allows to find start-oscillation current and frequency.

The method of solution of the TWT boundary problem can be easily generalized for the case of TWT consisting of several sections, for example, the TWT with local absorber. In that case, the transmission matrix should be represented as $\hat{T}_{TWT} = \hat{T}_1\hat{T}_2...\hat{T}_n$, where $\hat{T}_i$ — transmission matrix of *i*-th section.

4. Numerical results

A computer code for numerical solution is developed by means of the «Mathematica 6.0» software package. We perform calculations for TWT with coupled-cavity SWS (CC TWT). Using the results presented in [10-12] where nonstationary theory of the CC TWT has been developed, we obtain the following equations for phase velocity and coupling impedance dispersion

$$\beta_0(\omega) = \frac{\omega_0}{v_0} + \frac{\omega - \omega_0}{\Delta\omega d \sin\left[\arccos\left[-\frac{\omega - \omega_0}{\Delta\omega}\right]\right]},$$
$$K(\omega) = \frac{K(\omega_0)}{\sin\left[\arccos\left[-\frac{\omega - \omega_0}{\Delta\omega}\right]\right]}. \quad (28)$$

Here, $\Delta\omega$ and d denote the SWS bandwidth and period, respectively. It is necessary to emphasize that the developed theory is not applicable near the cutoff frequency, where group velocity becomes zero and coupling impedance becomes infinite.

First let us consider a single-section TWT with parameters listed in Table 1. The parameters are chosen close to that of the TWT described in [13], which had been widely used for satellite uplinks. Value K represented in Table 1 corresponds to the central frequency of passband, chosen accelerating voltage value corresponds to synchronism at the central frequency ω_0.

Table 1. TWT parameters

Beam current I_0, A	0.025
Beam voltage V_0, kV	15.32
Coupling impedance K, Ohm	38.93
Beam radius r_b, mm	1.25
Central frequency f_0, GHz	6.49
System period d, mm	5.9
Plasma frequency reduction factor R	0.48

In Fig. 2 gain versus normalized frequency is plotted. Dash-dot curve shows solution of the boundary value problem by method described above. The periodical structure is matched with the input and output waveguides exactly in the center of the passband, i.e. $\alpha_{0,l} = \omega_0 / v_0$ .The maximum gain about 16dB is obtained at a frequency close to synchronism. The results are in good agreement with the successive approximations method (dashed line) and dispersion equation method (solid line). However, the method of successive approximations agrees with the other methods only for the case of relatively small gain, less than 20 dB.

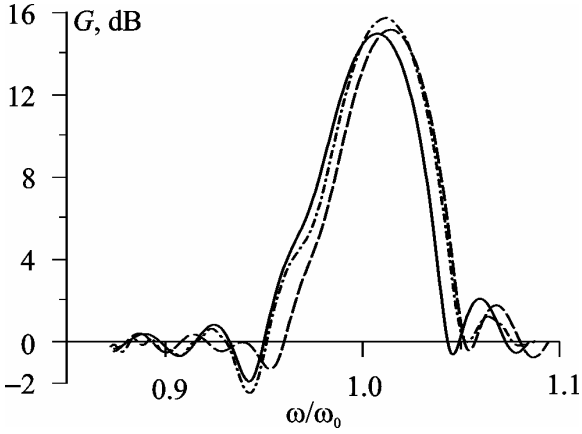

Fig. 2. TWT gain versus normalized frequency calculated using the successive approximations method (dashes), dispersion equation method (solid), and solution of the boundary problem (dash-dot)

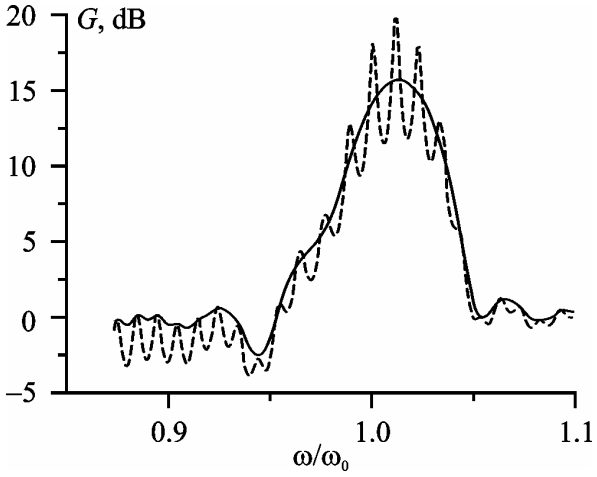

Fig. 3. TWT gain versus normalized frequency for different values of $\alpha_{0,l}$ parameters: $\alpha_{0,l} = \omega_0 / v_0$ (solid), $1.8\omega_0 / v_0$ (dashes)

When the match frequency moves off from the central frequency, the reflections grow. Fig. 3 shows $G(\omega)$ for $\alpha_{0,l} = 1.8\omega_0 / v_0$ compared with the curve for $\alpha_{0,l} = \omega_0 / v_0$. One can easily see strong gain ripples caused by the end reflections.

Fig. 4 shows field and beam current distributions along the system in case of matching at the central frequency when the DC beam current is increased up to 0.25 A. One can easily see the ripples on the curve $E(x)$ caused by reflections. Despite the input signal frequency is chosen equal to ω_0, reflections arise since "hot" propagation constant of the forward wave differs from the parameter α_l that is chosen equal to "cold" propagation constant ω_0 / v_0. To eliminate reflections α_l should be chosen approximately equal to real part of the propagation constant for the growing forward wave [9]. Fig. 5 shows field and beam current distribution along the system in case of matching at the frequency different from the input signal frequency. One can see considerable enhancement of reflections.

The main way to avoid TWT self-excitation caused by reflections is to introduce a local absorber. In that case, we have to solve a boundary problem for a multisectional TWT. Consider a TWT consisting of three sections where the second one is the absorber with attenuation $\gamma = 60$ dB/cm (Fig. 6a). Fig. 7 shows field and beam current distributions along the system for this case. Parameters $\alpha_{0,l}$ are chosen as $\alpha_{0,l} = \omega_0 / v_0$, i.e. matching frequency is equal to the central frequency, and the input signal frequency equals ω_0 as

well. One can see strong attenuation of the wave in the absorber section. Accordingly, in that domain bunching process is interrupted and the amplitude of the first harmonic of the beam current remains nearly constant. However, the rapid jump of the attenuation factor causes considerable reflections from the absorber. As a result, the $E(x)$ curve still exhibits strong ripples, especially in the first section located before the absorber.

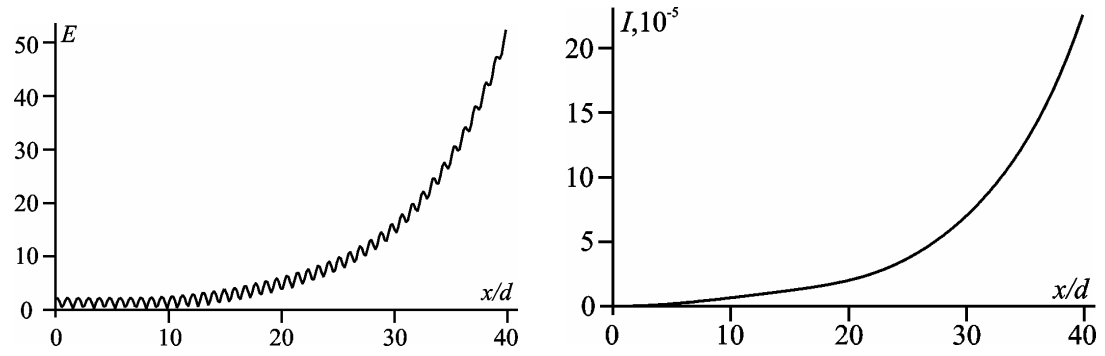

Fig. 4. Field and beam current distributions along the system for the single-section TWT in case of matching at the central frequency

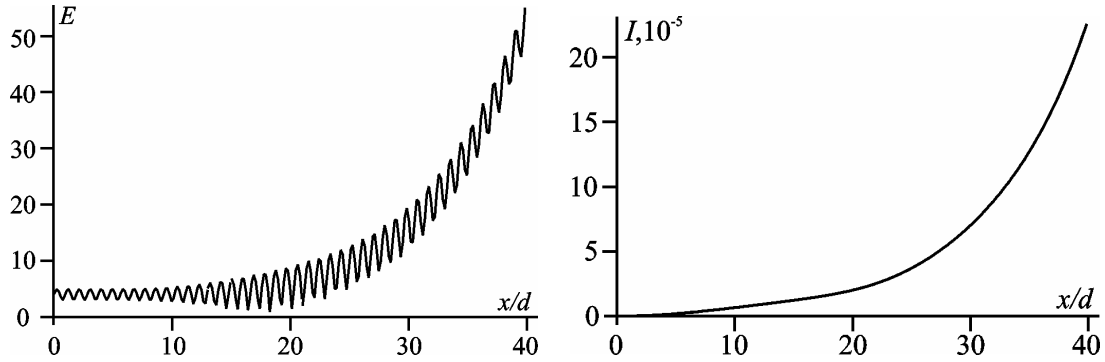

Fig. 5. Field and beam current distribution along the system for the single-section TWT in case of matching at the frequency $-0.9\Delta\omega + \omega_0$. No losses

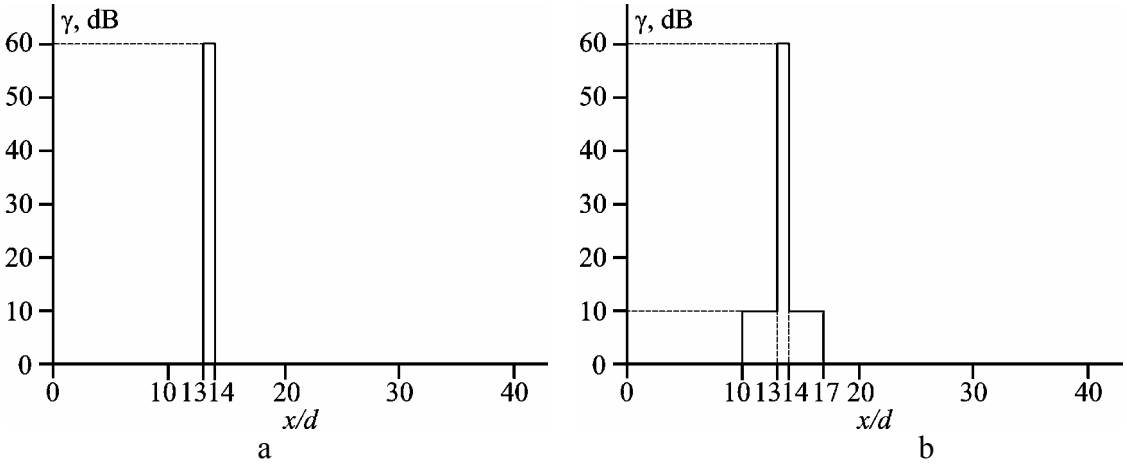

Fig. 6. Absorber profiles for TWT consisting of three (a) and five (b) sections

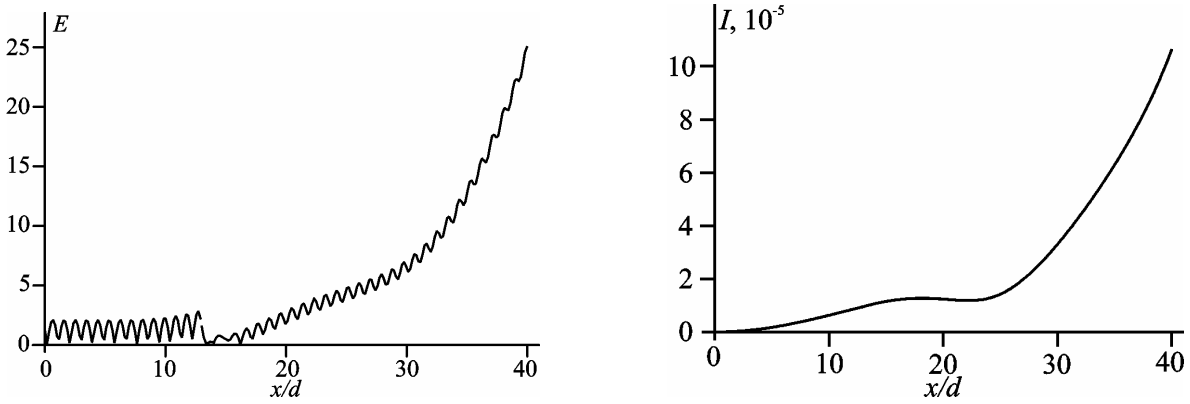

Fig. 7. Field and beam current distributions along the system for the TWT with absorber profile shown in Fig. 6a

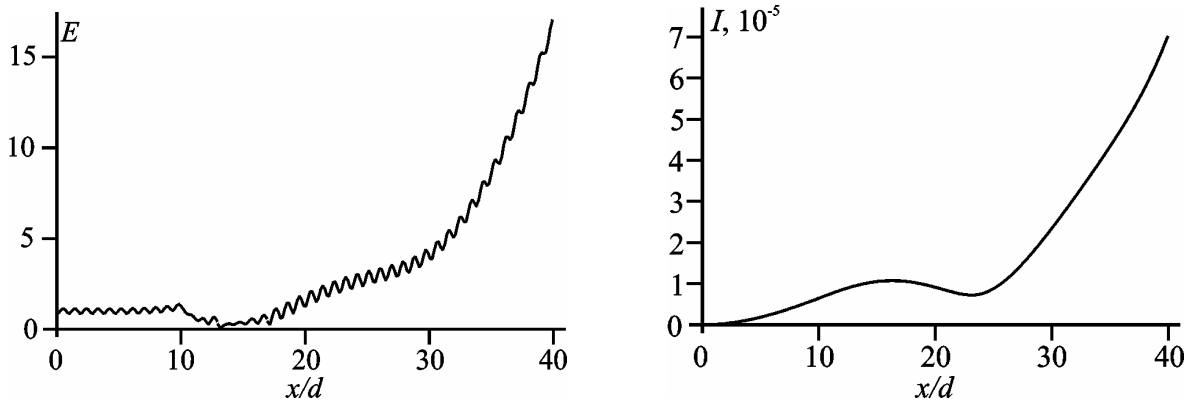

Fig. 8. Field and beam current distributions along the system for the TWT with matched absorber profile shown in Fig. 6b

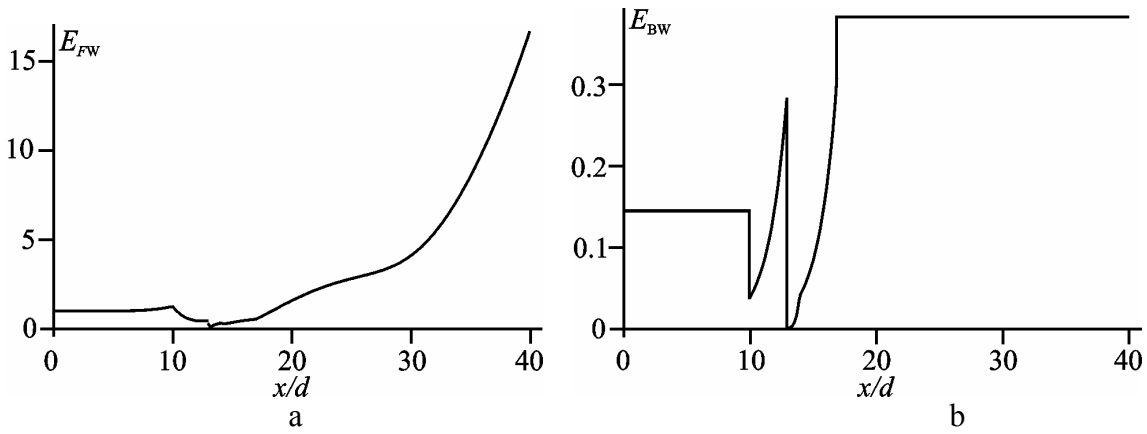

Fig. 9. Fields distributions for three forward waves (a) and backward wave (b) in the case of TWT with matched absorber

To avoid the reflections from the absorber, one should use tapering i.e. $\gamma(x)$ should be smoothly varying at the ends of the absorber. To simulate that case we add two matching sections with $\gamma = 10$ dB/cm attenuation, as is shown in Fig. 6b. Thus, we are dealing with

the TWT consisting of five sections. The field and beam current distributions along the tube plotted in Fig. 8 show substantial decrease of reflections. The gain value decreases as well.

The developed code allows calculation of field distributions separately for the three forward waves and the backward one. Such plots are shown in Fig. 9. From Fig. 9b one can see damping of the backward wave the absorber and its rapid jumps at the ends of the sections.

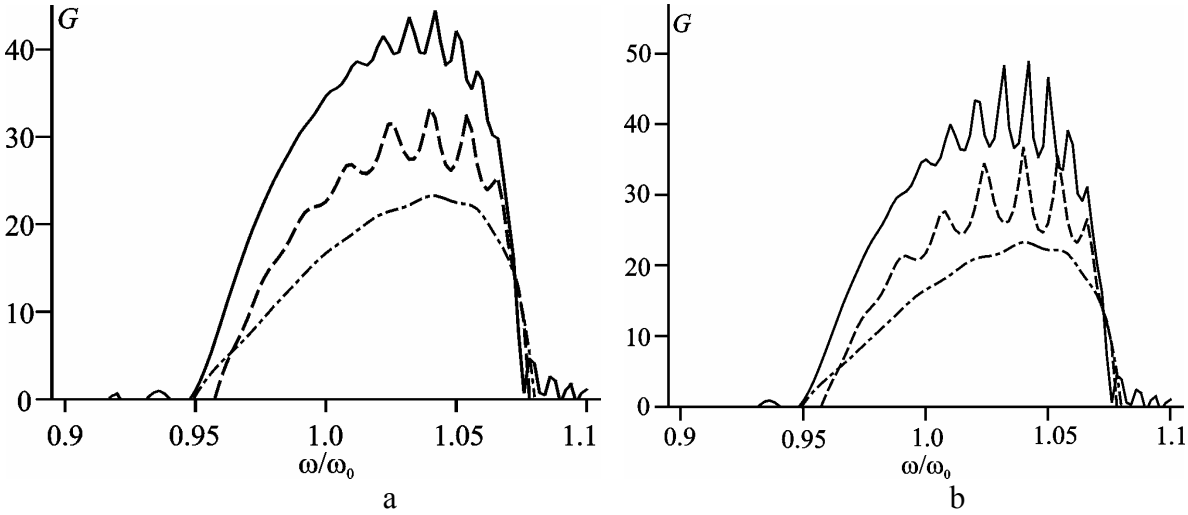

Fig 10. Gain vs. normalized frequency for the TWT without absorber (solid line) and with mismatched (dashes line) and matched absorber (dash-dot line). The match frequency is (a) equal to ω_0 and (b) $\omega_0 - \Delta\omega/2$

In Fig. 10a gain vs. frequency curves are is plotted for the case of matching exactly at the central frequency ω_0. In the case of TWT without absorber (solid line) strong gain ripples caused by reflections are observed. Introducing of the mismatched absorber that corresponds to $\gamma(x)$ shown in Fig. 6a (dashed line) does not reduce the ripples and results only in nearly 10 dB gain decrease. When the matching absorbing sections are added (dash-dot line), the ripples disappear, i.e. the reflections from the absorber are suppressed. When the match frequency is not equal to the central frequency the role of the absorber becomes more evident. In Fig 10b the curves similar to Fig. 10a are shown, but for matching at $\omega_0 - \Delta\omega/2$ frequency. The $G(\omega)$ ripples are stronger than on Fig. 10a, but when the absorber with matching sections is introduced, the gain ripples becomes suppressed and the dashed curves in Fig. 10a and Fig. 10b are almost identical.

5. Conclusion

Linear theory of a TWT amplifier with end reflections taken into account is developed in this article. The method for solution of the boundary problem using the Laplace transform is described. A computer code for numerical solution is developed by means of the «Mathematica 6.0» software package. The developed code plots both field and beam current distributions along the system at the different frequencies, and calculates gain versus frequency for TWT with arbitrary number of sections.

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CONTENTS

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